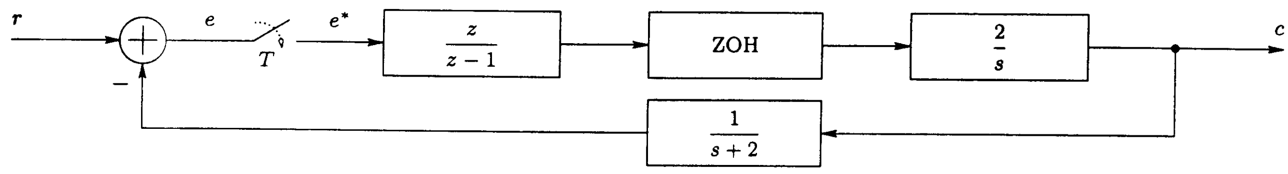
75 minutes

1. Consider the following system with a sampling period of 0.1 second.



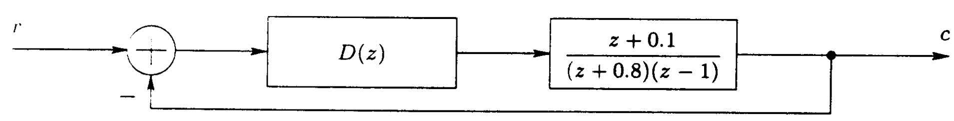
Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible. (25pts)

2. Consider a negative feedback discrete-time control system, where the loop gain is given by

$$G(z)H(z) = K\frac{z+0.5}{z^3 - z^2 - 0.25z + 0.25}.$$

Determine the range of stability in terms of the gain K. (25pts)

3. Consider the following system with a sampling period of 1 second.

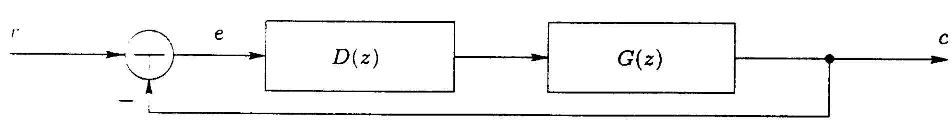


Design the simplest controller $D(z)$ that satisfies the following requirements. (25pts)

- The 5% settling-time is less than or equal to 3 seconds.
- The maximum percent-overshoot is between 15% and 25% for the unit-step input.
- The steady-state error is zero for a step input.

4. Consider the following feedback control system, where

$$G(z) = \frac{4950.5(z+1)(10001z+9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218}.$$



Determine the gain and phase margins of the closed-loop system. Design the simplest controller $D(z)$, such that gain margin of the system is increased by 100 dB. Assume the sampling period $T = 2$ s. (25pts)

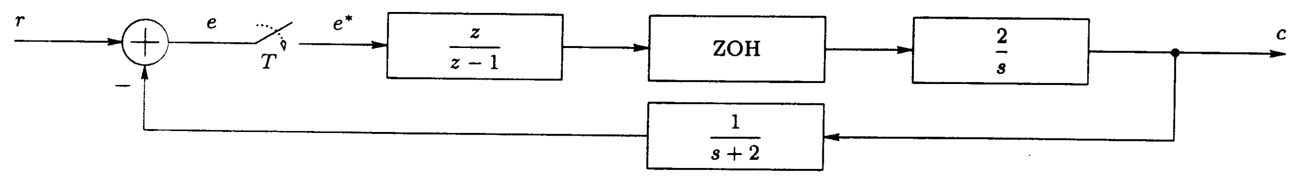
HINT:

$$\mathcal{W}\left[G\right](w) = \left[G(z)\right]_{z=\frac{1+(T/2)w}{1-(T/2)w}} = \left[\frac{4950.5(z+1)(10001z+9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218}\right]_{z=\frac{1+w}{1-w}} = \frac{5000(w+10000)^2}{w(w+0.01)(w+10)}.$$

EE 331 **Exam#1** **Feb. 28, 2006**

Solutions

1. Consider the following system with a sampling period of 0.1 second.



Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible.

Solution: In order to be able to take the z-transforms of signals, they need to be sampled or pseudo-sampled. Denoting the transfer function of the zero-order hold (ZOH) by G_{ZOH}, we have

$$E(s) = R(s) - \left(\frac{1}{s+2}\right) C(s) = R(s) - \left(\frac{1}{s+2}\right)\left(\frac{2}{s}\right) G^*_{\text{ZOH}}(s)\left(\frac{z}{z-1}\right) E^*(z),$$

where $E^*(z)$ represents the ideally-sampled $E(s)$. When we take the z-transforms of the inverse Laplace transforms in the above equation, we get

$$E(z) = R(z) - \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\frac{2}{s(s+2)}\right) G^*_{\text{ZOH}}(s)\right]\right](z)\left(\frac{z}{z-1}\right)E(z).$$

To simplify the notation, we let

$$\begin{aligned}
(G_{\text{ZOH}}G_1G_2)(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\frac{2}{s(s+2)}\right) G^*_{\text{ZOH}}(s)\right]\right](z) \\
&= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{2}{s^2(s+2)}\right]\right](z) \\
&= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[-\frac{1/2}{s} + \frac{1}{s^2} + \frac{1/2}{s+2}\right]\right](z) \\
&= \left(\frac{z-1}{z}\right)\left(-(1/2)\frac{z}{z-1} + \frac{Tz}{(z-1)^2} + (1/2)\frac{z}{z-e^{-2T}}\right) \\
&= \frac{(2T-1+e^{-2T})z + (1-(1+2T)e^{-2T})}{2(z-e^{-2T})(z-1)}.
\end{aligned}$$

Then,

$$E(z) = R(z) - \left(\frac{(2T-1+e^{-2T})z + (1-(1+2T)e^{-2T})}{2(z-e^{-2T})(z-1)}\right)\left(\frac{z}{z-1}\right)E(z),$$

or

$$\left(1 + \left(\frac{(2T-1+e^{-2T})z + (1-(1+2T)e^{-2T})}{2(z-e^{-2T})(z-1)}\right)\left(\frac{z}{z-1}\right)\right)E(z) = R(z);$$

and for $T = 0.1$ s, we get

$$\left(\frac{z^3 - 2.8094z^2 + 2.6462z - 0.8187}{(z-0.8187)(z-1)^2}\right) E(z) = R(z),$$

or

$$E(z) = \left(\frac{(z-0.8187)(z-1)^2}{z^3 - 2.8094z^2 + 2.6462z - 0.8187}\right) R(z).$$

The z-transform of the inverse Laplace transform on the pseudo-sampled output gives

$$\begin{aligned} C(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\frac{2}{s}\right) G^*_{\mathrm{ZOH}}(s)\right]\right](z)\left(\frac{z}{z-1}\right) E(z) \\ &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\frac{2}{s^2}\right)\right]\right](z)\left(\frac{z}{z-1}\right) E(z) \\ &= 2\frac{Tz}{(z-1)^2}E(z) = \frac{0.2z}{(z-1)^2}E(z). \end{aligned}$$

Substituting the expression for $E(z)$ in the previous equation, we get

$$C(z) = \left(\frac{0.2z}{(z-1)^2}\right)\left(\frac{(z-0.8187)(z-1)^2}{z^3 - 2.8094z^2 + 2.6462z - 0.8187}\right) R(z),$$

or

$$\frac{C(z)}{R(z)} = \frac{0.2z(z-0.8187)}{z^3 - 2.8094z^2 + 2.6462z - 0.8187}.$$

2. Consider a negative feedback discrete-time control system, where the loop gain is given by

$$G(z)H(z) = K\frac{z+0.5}{z^3 - z^2 - 0.25z + 0.25}.$$

Determine the range of stability in terms of the gain K.

Solution: For $G(z)H(z) = K\left((z+0.5)/(z^3 - z^2 - 0.25z + 0.25)\right)$, the characteristic equation is

$$1 + K\frac{z+0.5}{z^3 - z^2 - 0.25z + 0.25} = 0,$$

or

$$\frac{z^3 - z^2 - 0.25z + 0.25 + K(z+0.5)}{z^3 - z^2 - 0.25z + 0.25} = 0..$$

Therefore the characteristic polynomial is

$$q(z) = z^3 - z^2 + (K - 0.25)z + (0.5K + 0.25).$$

To determine the range of stability for all K, we can use Jury's stability test criteria. In our case, the order of the system $n = 3$. The two boundary conditions are

$$\begin{aligned} q(1) &> 0, \\ (1)^3 - (1)^2 + (K - 0.25)(1) + (0.5K + 0.25) &> 0, \\ K &> 0, \end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
(-1)^n q(-1) &> 0. \\
(-1)^3((-1)^3 - (-1)^2 + (K - 0.25)(-1) + (0.5K + 0.25)) &> 0, \\
K &> -3. \qquad (2.2)
\end{aligned}
$$

The pole-product condition is

$$
\begin{aligned}
|a_0| &< a_n, \\
|0.5K + 0.25| &< 1, \\
-1 < 0.5K + 0.25 &< 1, \\
-1.25 < 0.5K &< 0.75, \\
-2.5 < K &< 1.5. \qquad (2.3)
\end{aligned}
$$

The rest of the conditions is to be obtained from the Jury's table.

z^0	z^1	z^2	z^3
$a_0 = 0.5K + 0.25$	$a_1 = K - 0.25$	$a_2 = -1$	$a_3 = 1$
$a_3 = 1$	$a_2 = -1$	$a_1 = K - 0.25$	$a_0 = 0.5K + 0.25$
$a_0^1 = \det \begin{bmatrix} a_0 & a_3 \\ a_3 & a_0 \end{bmatrix}$ $= \det \begin{bmatrix} 0.5K + 0.25 & 1 \\ 1 & 0.5K + 0.25 \end{bmatrix}$ $a_0^1 = 0.25(K - 1.5)(K + 2.5)$	$a_1^1 = \det \begin{bmatrix} a_0 & a_2 \\ a_3 & a_1 \end{bmatrix}$	$a_2^1 = \det \begin{bmatrix} a_0 & a_1 \\ a_3 & a_2 \end{bmatrix}$ $= \det \begin{bmatrix} 0.5K + 0.25 & K - 0.25 \\ 1 & -1 \end{bmatrix}$ $a_2^1 = -1.5K$	

Since we have a third-order system, the table will only give one more additional condition.

$$
\begin{aligned}
|a_0^1| &> |a_{n-1}^1|, \\
|0.25(K - 1.5)(K + 2.5)| &> |-1.5K|, \\
|(K - 1.5)(K + 2.5)| &> 6|K|.
\end{aligned}
$$

From the Inequality 2.1, we know that $K > 0$, therefore we have

$$|(K - 1.5)(K + 2.5)| > 6K > 0.$$

$(K - 1.5)(K + 2.5) > 6K > 0$ Case:

In this case,

$$
\begin{aligned}
K^2 + K - 3.75 > 6K &> 0, \\
K^2 - 5K - 3.75 &> 0, \\
(K + 0.662278)(K - 5.662278) &> 0, \\
K < -0.662278, \text{ or } K &> 5.662278. \qquad (2.4)
\end{aligned}
$$

However in this case, the intersection of the regions described by Inequalities 2.1–2.3 and 2.4 is empty.

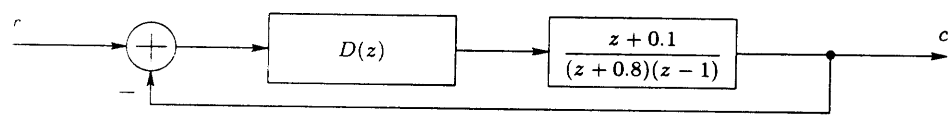
$-(K-1.5)(K+2.5) > 6K > 0$ Case:
In this case,

$$\begin{aligned}
-K^2 - K + 3.75 &> 6K > 0, \\
K^2 + 7K - 3.75 &< 0, \\
(K+7.5)(K-0.5) &< 0, \\
-7.5 < K &< 0.5. \qquad (2.5)
\end{aligned}$$

From the intersection of the regions described by Inequalities 2.1–2.3 and 2.5, we conclude that the system will be asymptotically stable, when

$$0 < K < 0.5.$$

3. Consider the following system with a sampling period of 1 second.



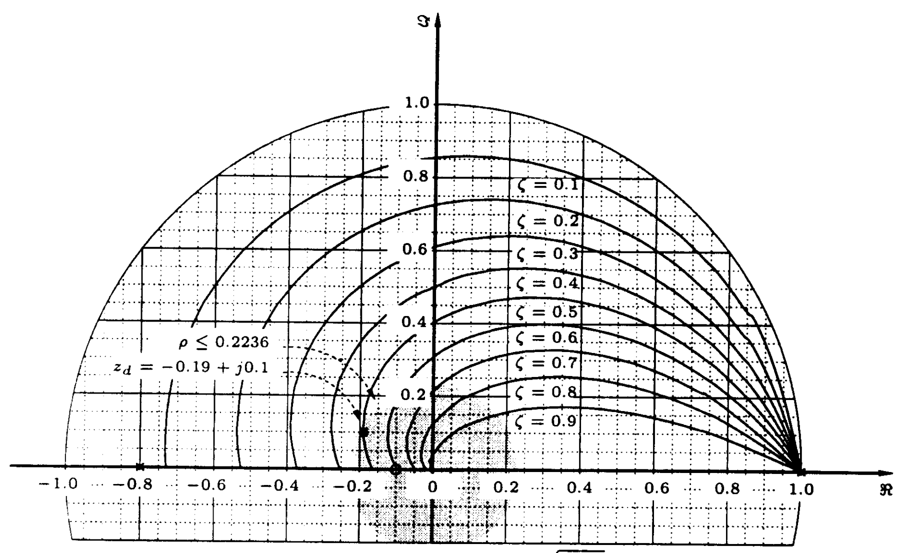
Design the simplest controller $D(z)$ that satisfies the following requirements.

- The 5% settling-time is less than or equal to 3 seconds.
- The maximum percent-overshoot is between 15% and 25% for the unit-step input.
- The steady-state error is zero for a step input.

Solution: We determine the restrictions on the location of the desired-pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent-overshoot for the unit-step input	$15\% \le M_p \le 25\%$.	From the α-M_p curves, $\zeta = 0.5$ provides the broadest range of α values.
Settling time for the unit-step input	$\rho \le (0.05)^{1/(k_{5\%_s}-1)}$.	For $t_{5\%_s} = k_{5\%_s}T \le 3\,\mathrm{s}$, and $k_{5\%_s} \le 3/1 = 3$, when $T = 1\,\mathrm{s}$; $\rho \le (0.05)^{1/(5-1)} = 0.2236$.
The steady-state error is is zero for a step input.	Open-loop gain has a pole at 1.	Open-loop gain $= D(z)\dfrac{z+0.1}{(z+0.8)(z-1)}$ has to have a pole at 1. Since the open-loop gain already has a pole at 1, as long as $D(z)$ doesn't cancel it, this requirement is satisfied.

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx -0.19 \pm j0.1$.



$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T} e^{j\sqrt{1-\zeta^2}\omega T}$$

The deficiency angle, ϕ, needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

$$\phi + \measuredangle(z_d - (-0.1)) - \measuredangle(z_d - (-0.8)) - \measuredangle(z_d - (1)) = (2k+1)\pi,$$

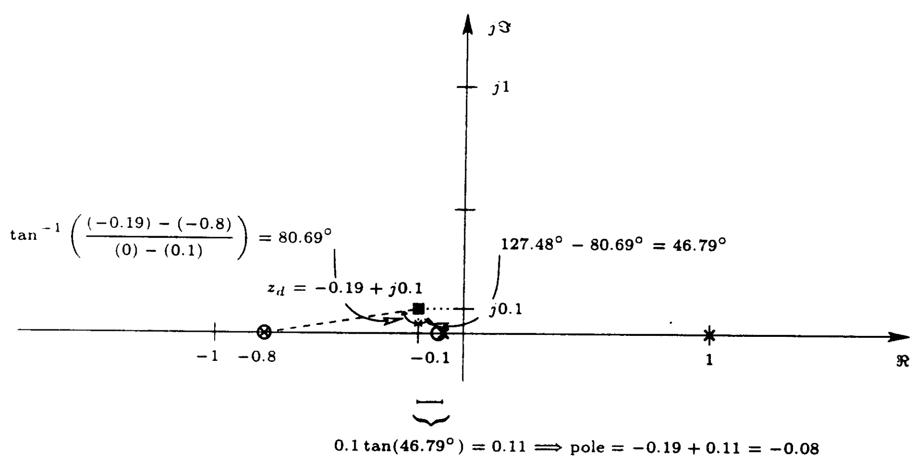
for an integer k. For $z_d = -0.19 + j0.1$,

$$\phi + \tan^{-1}\left(\frac{(0.1)-(0)}{(-0.19)-(-0.1)}\right) - \tan^{-1}\left(\frac{(0.1)-(0)}{(-0.19)-(-0.8)}\right) - \tan^{-1}\left(\frac{(0.1)-(0)}{(-0.19)-(1)}\right) = 180^\circ + k360^\circ,$$

$$\phi + 131.99^\circ - 9.31^\circ - 175.20^\circ = 180^\circ + k360^\circ,$$

or $\phi = -127.48^\circ$.

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at z_d. The best choice for cancellation is the pole at -0.8, since it is the slowest, and since canceling the other pole at 1 prevents the satisfaction of the steady-state requirement.



From the above analysis,

$$D(z) = K\frac{z+0.8}{z+0.08}.$$

And the magnitude K is obtained from the magnitude condition at z_d.

$$|D(z)G(z)|_{z=z_d} = 1,$$

$$\left|K\frac{z+0.1}{(z+0.08)(z-1)}\right|_{z=-0.19+j0.1} = 1,$$

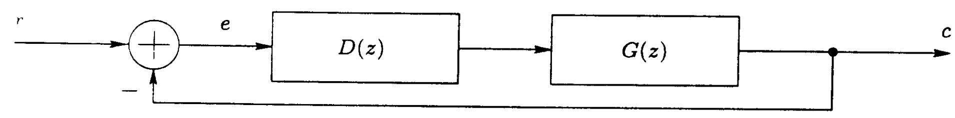
or $K = 1.3196$. Therefore,

$$D(z) = 1.3196\frac{z+0.8}{z+0.08}$$

is one possible controller.

4. Consider the following feedback control system, where

$$G(z) = \frac{4950.5(z+1)(10001z+9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218}.$$



Determine the gain and phase margins of the closed-loop system. Design the simplest controller $D(z)$, such that gain margin of the system is increased by 100 dB. Assume the sampling period $T = 2\,\mathrm{s}$.

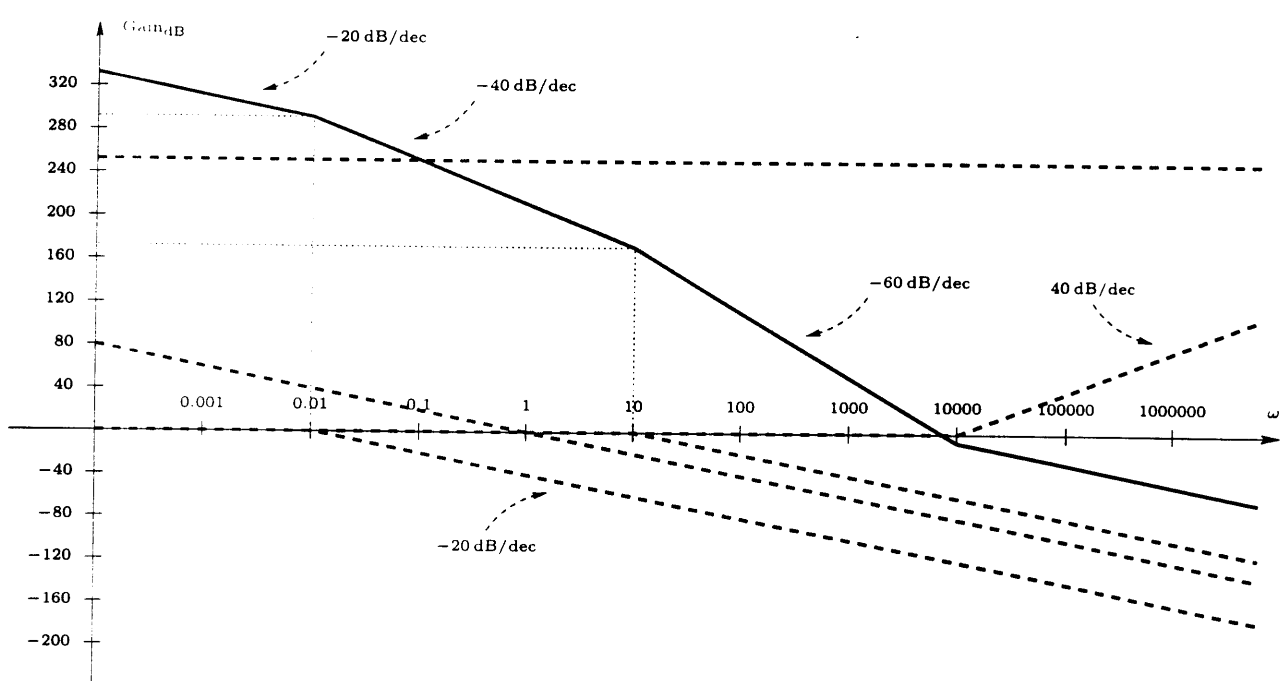
HINT:

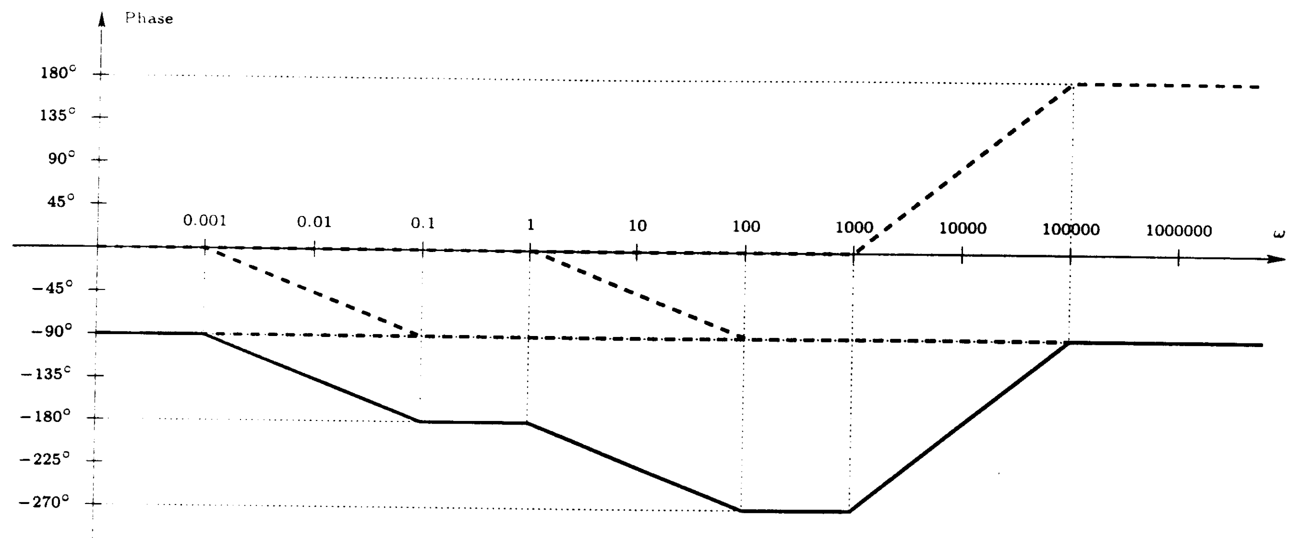
$$\mathcal{W}\left[G\right](w) = \left[G(z)\right]_{z=\frac{1+(T/2)w}{1-(T/2)w}} = \left[\frac{4950.5(z+1)(10001z+9999)^2}{11z^3 - 12.7822z^2 - 7.0396z + 8.8218}\right]_{z=\frac{1+w}{1-w}} = \frac{5000(w+10000)^2}{w(w+0.01)(w+10)}.$$

Solution: From the w-transform of G

$$\mathcal{W}\left[G\right](w) = \frac{5000(w+10000)^2}{w(w+0.01)(w+10)} = \frac{5\times10^{12}(1+w/10000)^2}{w(1+w/0.01)(1+w/10)},$$

we determine the cut-off frequencies and the $\mathrm{Gain_{dB}} = 20\log(5\times10^{12})\,\mathrm{dB} = 253.98\,\mathrm{dB}$ to plot the asymptotic bode plots.





In order to determine the gain and phase margins, we first need to determine the phase and the gain-crossover frequencies.

The phase-crossover frequency is determined from the bode plots, when phase angle becomes $-180°$ for the first time. From the asymptotic phase-bode plot, we observe that the $-180°$ crossing is at the mid point of $\omega = 0.1$ and $\omega = 1$. Since the horizontal scale is logarithmic, the mid point is such that

$$\log(\omega_p) = \frac{1}{2}\Big(\log(0.1) + \log(1)\Big) = -\frac{1}{2},$$

or

$$\omega_p = 10^{-1/2} \approx 0.32\,\mathrm{rad/s}.$$

At the mid point of $\omega = 0.1$ and $\omega = 1$, the gain is approximately $-20\,\mathrm{dB} + 253.98\,\mathrm{dB} = 233.98\,\mathrm{dB}$. As a result, the gain margin is approximately $-233.98\,\mathrm{dB}$.

The gain-cross-over frequency is determined from the bode plots, when the gain is $0\,\mathrm{dB}$ for the first time. From the asymptotic gain-bode plot, we observe that the $0\,\mathrm{dB}$ gain is between $\omega = 1000$ and $\omega = 10000$. At $\omega = 1000$, the gain is $-200\,\mathrm{dB} + 253.98\,\mathrm{dB} = 53.98\,\mathrm{dB}$; and at $\omega = 10000$, the gain is $-260\,\mathrm{dB} + 253.98\,\mathrm{dB} = -6.02\,\mathrm{dB}$. So, using the straight-line equation, we get the gain-cross-over frequency, such that

$$\log(\omega_g) - \log(1000) = \left(\frac{\log(10000) - \log(1000)}{(-6.02) - (53.98)}\right)\Big((0) - (53.98)\Big) = \frac{53.98}{60},$$

or

$$\omega_g = 10^{53.98/60+3} \approx 7937\,\mathrm{rad/s}.$$

Again using the straight-line approximation, we get

$$\mathrm{Phase}\Big|_{\omega=7937} - (-270°) = \left(\frac{(-180°) - (-270°)}{\log(10000) - \log(1000)}\right)\Big(\log(7937) - \log(1000)\Big) = 80.97°,$$

or

$$\mathrm{Phase}\Big|_{\omega=7937} = -189.03°.$$

As a result, the phase margin is approximately $-189.03° + 180° = -9.03°$.

The system is unstable, since the gain margin is $-233.98\,\text{dB}$, and the phase margin is $-9.03°$.

We may proceed to design the desired compensator as a lead or a lag compensator. However, in this case, the lag compensator design is a lot simpler; since all we need to do is to supply $-100\,\text{dB}$ gain at a low enough frequency, so that the phase-angle of the lag compensator does not interfere with the phase cross-over frequency. The lag compensator is given by

$$\mathcal{W}\left[D\right](j\omega) = \frac{1 + j\omega/\omega_L}{1 + j\omega/(\omega_L/\beta)}.$$

Choosing the cut-off frequency ω_L of the compensator at least 10 times slower than the gain-crossover frequency $\omega_g = 0.32\,\text{rad/s}$, so that the negative phase angle of the compensator doesn't affect the phase angle directly, we get the lag compensator in the w-transform domain as

$$\mathcal{W}\left[D\right](j\omega) = \frac{1 + j\omega/0.01}{1 + j\omega/(0.01/\beta)}.$$

Since this gain is to be reduced by 100 dB by the lag compensator, the compensator gain β is such that

$$|\beta|_{\text{dB}} = 20\log(\beta) = 100,$$

or $\beta = 10^5$. However, we need to remember that this value of β is too large to be practically feasible.

Finally, $D(z)$ is determined from

$$D(z) = \left[\mathcal{W}\left[D\right](w)\right]_{w=\frac{2}{T}\frac{z-1}{z+1}} = \left[\frac{1 + w/10^{-2}}{1 + w/10^{-7}}\right]_{w=\frac{z-1}{z+1}},$$

or

$$D(z) = 10^{-5}\left(\frac{z - 0.98}{z - 1}\right).$$