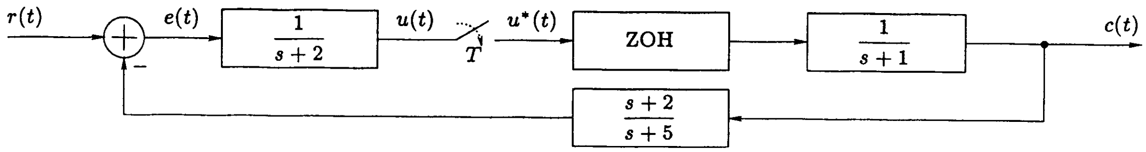
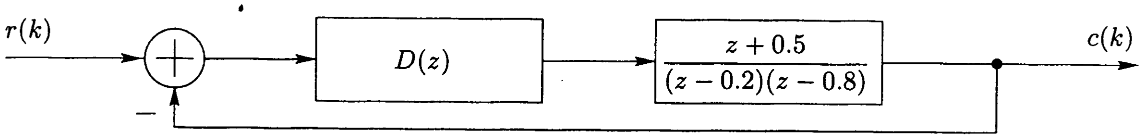
EE 331 | **Exam#1** | **Mar. 08, 2001**

75 minutes

1. Consider the following system with a sampling period $T = 0.1$ second. Obtain an expression for the transfer function $C(z)/R(z)$. (50pts)



2. Consider the following feedback control system with a sampling period $T = 0.1$ second.



(a) Assume a PI controller in the form

$$D(z) = K_1 + \frac{K_2}{z-1}.$$

Choose K_2 (in terms of K_1), such that the slowest open-loop pole is canceled, and determine the range of stability for all K_1 with this choice of K_2. (20pts)

(b) Design a first-order compensator such that

$$D(z) = K\frac{z - z_1}{z - p_1},$$

where K, z_1, and p_1 are the gain, the zero, and the pole of the compensator, respectively, such that the following conditions are satisfied.

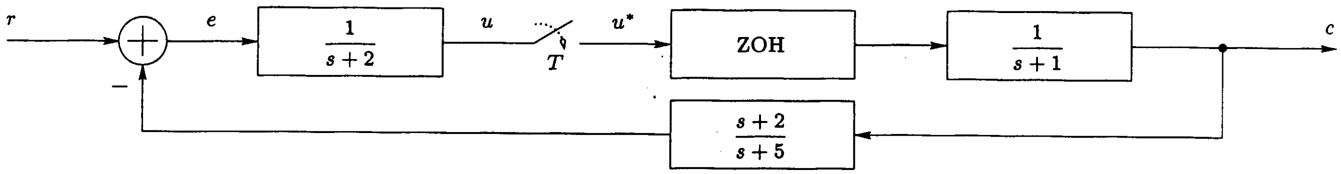
i. The maximum percent overshoot is between 1% and 3% for a unit step input.

ii. The 2% settling time is less than 0.4 second.

(30pts)

EE 331 | **Exam#1 Solutions** | Mar. 08, 2001

1. Consider the following system with a sampling period $T = 0.1$ second. Obtain an expression for the transfer function $C(z)/R(z)$.



Solution: Assuming that there are pseudo samplers at the input and the output, we write the relationships among pseudo sampled signals.

$$C^*(s) = \frac{1}{s+1} G_{\text{ZOH}}(s) U^*(s),$$

where $(\cdot)^*$ represents a pseudo sampled signal, and $G_{\text{ZOH}}(s)$ is the transfer function of the zero-order hold (ZOH). So,

$$C(z) = \frac{z-1}{z} \mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right]\right] U(z).$$

Also

$$U^*(s) = \frac{1}{s+2}\left(R^*(s) - \left(\frac{s+2}{s+5}\right)\left(\frac{1}{s+1}\right) G_{\text{ZOH}}(s) U^*(s)\right),$$

or

$$\left(1 + \frac{1}{(s+1)(s+5)} G_{\text{ZOH}}(s)\right) U^*(s) = \frac{1}{s+2} R^*(s),$$

and

$$\left(\mathcal{Z}\left[\mathcal{L}^{-1}[1]\right] + \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]\right]\right) U(z) = \mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right] R(z).$$

Solving for $U(z)$ from the above equation and substituting into the expression for $C(z)$, we get

$$\frac{C(z)}{R(z)} = \frac{\frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right]\right]\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right]}{1 + \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]\right]}.$$

Next, we need to determine all the z transforms of the inverse Laplace transforms. So consider

$$\begin{aligned}
\frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right]\right] &= \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] \\
&= 1 - \frac{z-1}{z-e^{-T}} = \frac{1-e^{-T}}{z-e^{-T}} \approx \frac{0.0952}{z-0.9048},
\end{aligned}$$

$$\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s+2}\right]\right] = \frac{z}{z-e^{-2T}} \approx \frac{z}{z-0.8187},$$

and

$$\begin{aligned}
\frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1}{s(s+1)(s+5)}\right]\right] &= \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left[\frac{1/5}{s} - \frac{1/4}{s+1} + \frac{1/20}{s+5}\right]\right] \\
&= \frac{z-1}{z}\left[(1/5)\frac{z}{z-1} - (1/4)\frac{z}{z-e^{-T}} + (1/20)\frac{z}{z-e^{-5T}}\right] \\
&= \frac{1}{5} - \frac{z-1}{4(z-e^{-T})} + \frac{z-1}{20(z-e^{-5T})} \\
&= \frac{1}{5} - \frac{(z-1)(4z-(5e^{-5T}-e^{-T}))}{20(z-e^{-T})(z-e^{-5T})} \\
&\approx \frac{1}{5} - \frac{(z-1)(4z-2.1278)}{20(z-0.9048)(z-0.6065)}.
\end{aligned}$$

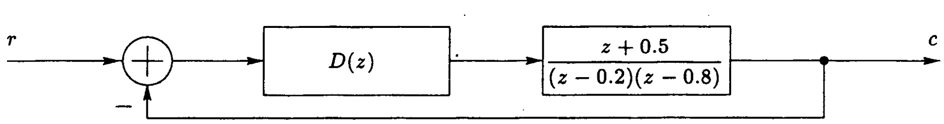
Substituting all the expressions into the $C(z)/R(z)$ expression gives

$$\frac{C(z)}{R(z)} = \frac{\left(\frac{1-e^{-T}}{z-e^{-T}}\right)\left(\frac{z}{z-e^{-2T}}\right)}{1+\left(\frac{1}{5} - \frac{(z-1)(4z-(5e^{-5T}-e^{-T}))}{20(z-e^{-T})(z-e^{-5T})}\right)}.$$

After rearranging some of the expressions above, we have

$$\begin{aligned}
\frac{C(z)}{R(z)} &= \frac{20(1-e^{-T})z(z-e^{-5T})}{\left(24(z-e^{-T})(z-e^{-5T}) - (z-1)(4z-(5e^{-5T}-e^{-T}))\right)(z-e^{-2T})} \\
&\approx \frac{1.9033z(z-0.6065)}{\left(24(z-0.9048)(z-0.6065) - (z-1)(4z-2.1278)\right)(z-0.8187)}.
\end{aligned}$$

2. Consider the following feedback control system with a sampling period $T = 0.1$ second.



(a) Assume a PI controller in the form

$$D(z) = K_1 + \frac{K_2}{z-1}.$$

Choose K_2 (in terms of K_1), such that the slowest open-loop pole is canceled, and determine the range of stability for all K_1 with this choice of K_2.

Solution: Rewriting the controller, we have

$$D(z) = K_1 + \frac{K_2}{z-1} = \frac{K_1(z-1)+K_2}{z-1} = K_1\frac{z-(1-K_2/K_1)}{z-1}.$$

Since the slowest pole is at 0.8, we choose the zero of the controller, such that

$$1 - K_2/K_1 = 0.8,$$

or

$$K_2 = 0.2K_1.$$

For this choice, the controller becomes

$$D(z) = K_1\frac{z-0.8}{z-1}.$$

The poles of the closed-loop system can be determined from the solution of the characteristic equation $1 + D(z)G(z) = 0$, where $G(z)$ is the plant transfer function, so

$$(z-0.2)(z-1) + K_1(z+0.5) = 0,$$
$$z^2 + (K_1 - 1.2)z + (0.5K_1 + 0.2) = 0.$$

To determine the range of stability for all K_1, we can use Jury's stability test criteria. Since the characteristic polynomial $q(z) = z^2 + (K_1 - 1.2)z + (0.5K_1 + 0.2)$, the order of the system $n = 2$. The two boundary conditions are

$$q(1) > 0,$$
$$(1)^2 + (K_1 - 1.2)(1) + (0.5K_1 + 0.2) > 0,$$
$$K_1 > 0, \tag{2.1}$$

and

$$(-1)^n q(-1) > 0,$$
$$(-1)^2\big((-1)^2 + (K_1 - 1.2)(-1) + (0.5K_1 + 0.2)\big) > 0,$$
$$K_1 < 4.8. \tag{2.2}$$

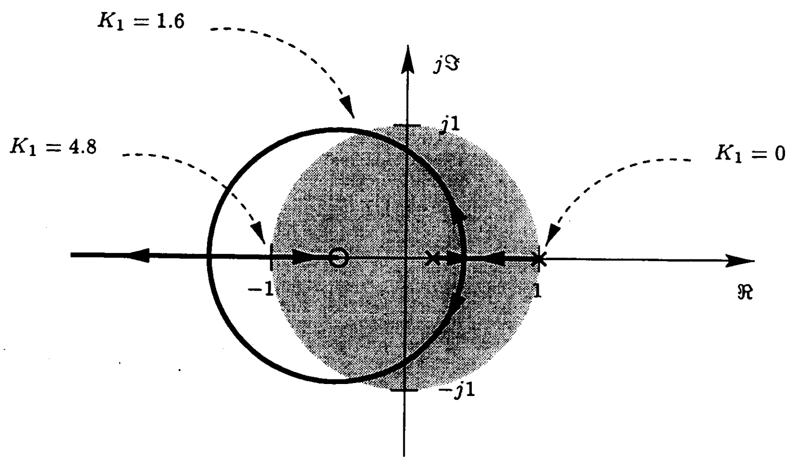
The pole-product condition is

$$|a_0| < a_n,$$
$$|0.5K_1 + 0.2| < 1,$$
$$-1 < 0.5K_1 + 0.2 < 1,$$
$$-2.4 < K_1 < 1.6. \tag{2.3}$$

The rest of the conditions are to be obtained from the Jury's table. However, since we have a second-order system, the table will not give any additional conditions. From the intersection of the regions described by Inequalities 2.1–2.3, we conclude that the system will be stable, when

$$0 < K_1 < 1.6.$$

The regions described by Inequalities 2.1–2.3 can also be observed from the root-locus diagram of the system for $K_1 \geq 0$.



(b) Design a first-order compensator such that

$$D(z) = K\frac{z - z_1}{z - p_1},$$

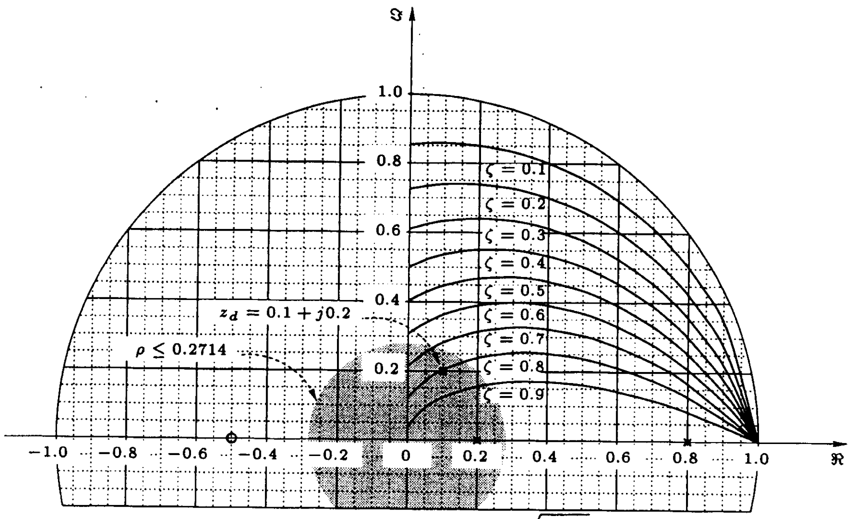
where K, z_1, and p_1 are the gain, the zero, and the pole of the compensator, respectively, such that the following conditions are satisfied.

i. The maximum percent overshoot is between 1% and 3% for a unit step input.
ii. The 2% settling time is less than 0.4 second.

Solution: We determine the restrictions on the location of the desired pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent overshoot for a unit step input	$0.01 < M_p < 0.03$.	From the α-M_p curves, $\zeta = 0.8$ provides the broadest range of α values, where $-80° < \alpha < 5°$.
Settling time for a unit step input	$\rho \le (0.02)^{1/(k_{2\%s}-1)}$.	For $t_{2\%s} = k_{2\%s}T \le 0.4\,\text{s}$, and $k_{2\%s} \le 0.4/0.1 = 4$, when $T = 0.1\,\text{s}$; $\rho \le (0.02)^{1/(4-1)} = 0.2714$.

When we mark these restrictions on the z-plane, we determine that a possible set of desired pole locations is at $z_d \approx 0.1 \pm j0.2$.



$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T}e^{j\sqrt{1-\zeta^2}\omega T}$$

The deficiency angle, ϕ, needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

$$\phi + \measuredangle(z_d - (-0.5)) - \measuredangle(z_d - (0.2)) - \measuredangle(z_d - (0.8)) = (2k+1)\pi,$$

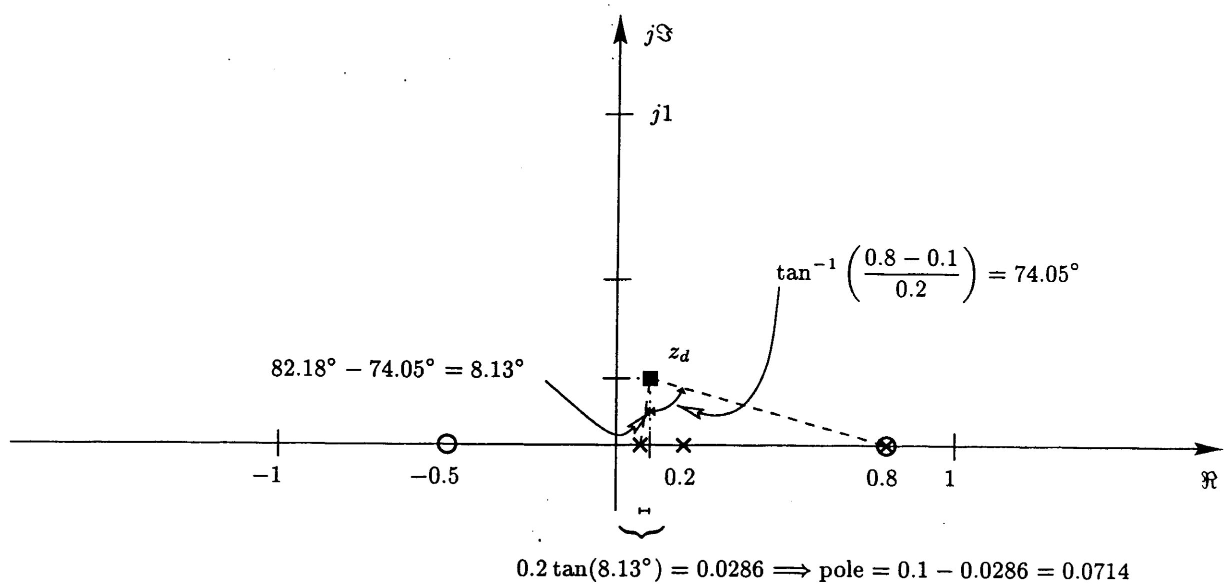
for an integer k. For $z_d = 0.1 + j0.2$,

$$\phi + \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(-0.5)}\right) - \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(0.2)}\right) - \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(0.8)}\right) = 180^\circ + k360^\circ,$$

$$\phi + 18.43^\circ - 116.57^\circ - 164.05^\circ = 180^\circ + k360^\circ,$$

or $\phi = 82.18^\circ$.

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at z_d. The best choice for cancelation is the pole at 0.8, since it is the slowest.



From the above analysis,

$$D(z) = K\frac{z-0.8}{z-0.0714}.$$

And the magnitude K is obtained from the magnitude condition at z_d.

$$\left|D(z)G(z)\right|_{z=z_d} = 1,$$

$$\left|K\frac{z+0.5}{(z-0.0714)(z-0.2)}\right|_{z=0.1+j0.2} = 1,$$

or $K = 0.0714$. Therefore, one possible compensator is

$$D(z) = 0.0714\frac{z-0.8}{z-0.0714}.$$