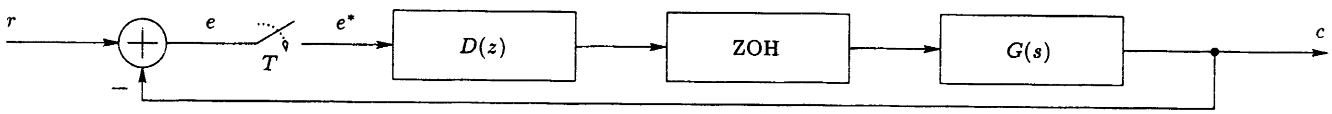
EE 331 **Exam#1** **Mar. 06, 2003**

75 minutes

1. Consider the following negative unity-feedback control system,



where

$$G(s) = 1 - \frac{s(s+1.44)}{(s+0.44)^2+1},$$

and $T = (\pi/4)$ s.

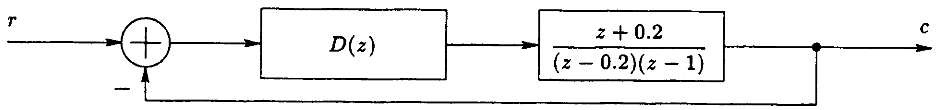
(a) Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible. (20pts)

(b) Assume that an integral controller

$$D(z) = K\frac{z}{z-1}$$

needs to be designed. Determine the range of stability in terms of the controller gain K. (15pts)

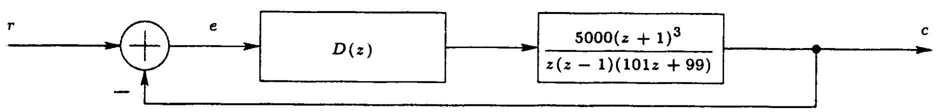
2. A controller for the following negative unity-feedback system with $T = 0.5$ s is to be designed.



(a) Design the simplest controller $D(z)$ that satisfies the following requirements. (20pts)
   - The steady-state error is zero for a unit-step input.
   - Maximum percent overshoot is $5\% \pm 1\%$ for a unit-step input.
   - The 2% settling time is less than 2.5 seconds.

(b) Design another controller cascaded to the first one, such that the steady-state error for the unit-ramp input is reduced to one fourth of its value, and the location of the existing closed-loop poles stay approximately the same. (15pts)

3. Consider the following feedback control system.



Design the simplest controller $D(z)$, such that the steady-state error for a unit-ramp input $e(\infty) \leq 0.01$, and the gain margin of the system is about 20 dB. Assume the sampling period $T = 2$ s. (30pts)

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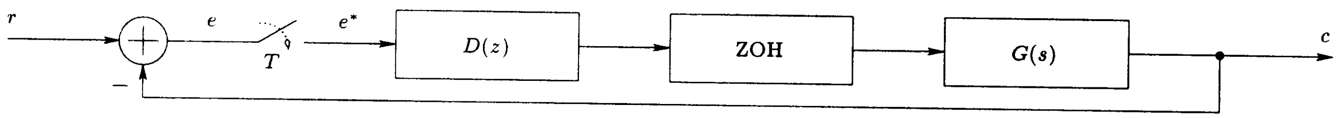
Solutions

1. Consider the following negative unity-feedback control system,



where

$$G(s) = 1 - \frac{s(s+1.44)}{(s+0.44)^2+1},$$

and $T = (\pi/4)$ s.

(a) Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible.

Solution: Assuming that there are pseudo samplers at the input and the output, we write the relationships among the pseudo sampled signals.

$$C^*(s) = G(s)G_{\mathrm{ZOH}}(s)D^*(s)E^*(s),$$

where $(\cdot)^*$ represents a pseudo-sampled signal, $G_{\mathrm{ZOH}}(s)$ is the transfer function of the zero-order hold (ZOH), and $D^*(s)$ is the Laplace transform of the gain D. So,

$$C(z) = \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{G(s)}{s}\right]\right](z)D(z)E(z).$$

Also

$$E(z) = R(z) - C(z).$$

Substituting $U(z)$ into the expression for $C(z)$, we get

$$\frac{C(z)}{R(z)} = \frac{((z-1)/z)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[G(s)/s\right]\right](z)D(z)}{1+((z-1)/z)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[G(s)/s\right]\right](z)D(z)}.$$

Next, we need to determine the z transform of the inverse Laplace transform. So consider

$$\begin{aligned}
\left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{G(s)}{s}\right]\right](z) &= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{1}{s} - \frac{s+1.44}{(s+0.44)^2+1}\right]\right](z)\\
&= \left(\frac{z-1}{z}\right)\mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{1}{s} - \frac{s+0.44}{(s+0.44)^2+1} - \frac{1}{(s+0.44)^2+1}\right]\right](z)\\
&= \left(\frac{z-1}{z}\right)\left[\frac{z}{z-1} - \frac{z^2-0.5z}{z^2-z+0.5} - \frac{0.5z}{z^2-z+0.5}\right]\\
&= \left(\frac{z-1}{z}\right)\left[\frac{z}{z-1} - \frac{z^2}{z^2-z+0.5}\right]\\
&= 1 - \frac{z(z-1)}{z^2-z+0.5} = \frac{0.5}{z^2-z+0.5}.
\end{aligned}$$

Since, from the transform pairs

$$e^{-\alpha t}\sin(\omega t)\mathbb{1}(t) \longleftrightarrow \frac{\omega}{(s+\alpha)^2+\omega^2},$$

$$e^{-\alpha t}\cos(\omega t)\mathbb{1}(t) \longleftrightarrow \frac{s+\alpha}{(s+\alpha)^2+\omega^2};$$

we have the pairs

$$e^{-0.44t}\sin(t)\mathbb{1}(t) \longleftrightarrow \frac{1}{(s+0.44)^2+1},$$

$$e^{-0.44t}\cos(t)\mathbb{1}(t) \longleftrightarrow \frac{s+0.44}{(s+0.44)^2+1}.$$

Moreover, since $t = kT = (\pi/4)k$, and the transform pairs

$$a^k\sin(bk)\mathbb{1}(k) \longleftrightarrow \frac{a\sin(b)z}{z^2-2a\cos(b)z+a^2},$$

$$a^k\cos(bk)\mathbb{1}(k) \longleftrightarrow \frac{z\big(z-a\cos(b)\big)}{z^2-2a\cos(b)z+a^2};$$

we have the pairs

$$\big(e^{-0.44(\pi/4)}\big)^k\sin\big((\pi/4)k\big)\mathbb{1}(k)$$

$$\longleftrightarrow \frac{e^{-0.44(\pi/4)}\sin\big(\pi/4\big)z}{z^2-2e^{-0.44(\pi/4)}\cos\big(\pi/4\big)z+\big(e^{-0.44(\pi/4)}\big)^2} = \frac{0.5z}{z^2-z+0.5},$$

$$\big(e^{-0.44(\pi/4)}\big)^k\cos\big((\pi/4)k\big)\mathbb{1}(k)$$

$$\longleftrightarrow \frac{z^2-e^{-0.44(\pi/4)}\cos\big(\pi/4\big)z}{z^2-2e^{-0.44(\pi/4)}\cos\big(\pi/4\big)z+\big(e^{-0.44(\pi/4)}\big)^2} = \frac{z^2-0.5z}{z^2-z+0.5}.$$

Substituting the z-transform expression into the $C(z)/R(z)$ expression gives

$$\frac{C(z)}{R(z)} = \frac{0.5D(z)}{z^2-z+0.5+0.5D(z)}.$$

(b) Assume that an integral controller

$$D(z) = K\frac{z}{z-1}$$

needs to be designed. Determine the range of stability in terms of the controller gain K.

Solution: For $D(z) = K\big(z/(z-1)\big)$, the characteristic equation is

$$z^2-z+0.5+0.5K\left(\frac{z}{z-1}\right) = 0,$$

or

$$\frac{z^3 - 2z^2 + 0.5(K+3)z - 0.5}{z-1} = 0.$$

Therefore the characteristic polynomial is

$$q(z) = z^3 - 2z^2 + 0.5(K+3)z - 0.5.$$

To determine the range of stability for all K, we can use Jury's stability test criteria. In our case, the order of the system $n = 3$. The two boundary conditions are

$$\begin{gathered} q(1) > 0, \\ (1)^3 - 2(1)^2 + 0.5(K+3)(1) - 0.5 > 0, \\ K > 0, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} (-1)^n q(-1) > 0, \\ (-1)^3\left((-1)^3 - 2(-1)^2 + 0.5(K+3)(-1) - 0.5\right) > 0, \\ K < 10. \end{gathered} \tag{1.2}$$

The pole-product condition is

$$\begin{gathered} |a_0| < a_n, \\ |-0.5| < 1, \end{gathered}$$

and it is satisfied independent of K.

The rest of the conditions is to be obtained from the Jury's table.

z^0	z^1	z^2	z^3
$a_0 = -0.5$	$a_1 = 0.5(K+3)$	$a_2 = -2$	$a_3 = 1$
$a_3 = 1$	$a_2 = -2$	$a_1 = 0.5(K+3)$	$a_0 = -0.5$
$a_0^1 = \det\begin{bmatrix} a_0 & a_3 \\ a_3 & a_0 \end{bmatrix} = \det\begin{bmatrix} -0.5 & 1 \\ 1 & -0.5 \end{bmatrix}$, $a_0^1 = -0.75$	$a_1^1 = \det\begin{bmatrix} a_0 & a_2 \\ a_3 & a_1 \end{bmatrix}$	$a_2^1 = \det\begin{bmatrix} a_0 & a_1 \\ a_3 & a_2 \end{bmatrix} = \det\begin{bmatrix} -0.5 & 0.5(K+3) \\ 1 & -2 \end{bmatrix}$, $a_2^1 = -0.5K - 0.5$	

Since we have a third-order system, the table will only give one more additional condition.

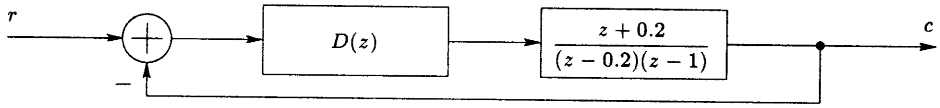
$$\begin{gathered} |a_0^1| > |a_{n-1}^1|, \\ |-0.75| > |-0.5K - 0.5|, \\ -0.75 < -0.5K - 0.5 < 0.75, \\ -0.25 < -0.5K < 1.25, \\ 0.5 > K > -2.5, \end{gathered}$$

$$-2.5 < K < 0.5. \tag{1.3}$$

From the intersection of the regions described by Inequalities 1.1–1.3, we conclude that the system will be asymptotically stable, when

$$0 < K < 0.5.$$

2. A controller for the following negative unity-feedback system with $T = 0.5\,\text{s}$ is to be designed.

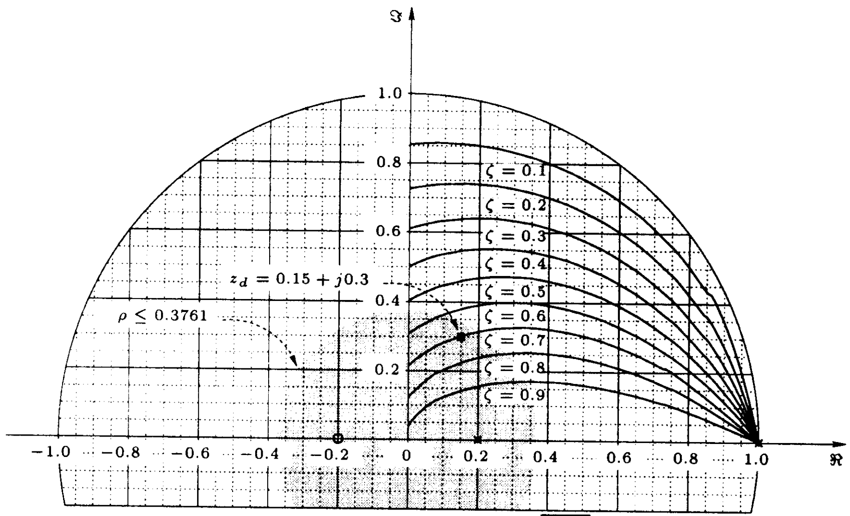


(a) Design the simplest controller $D(z)$ that satisfies the following requirements.
   - The steady-state error is zero for a unit-step input.
   - Maximum percent overshoot is $5\% \pm 1\%$ for a unit-step input.
   - The 2% settling time is less than 2.5 seconds.

Solution: We determine the restrictions on the location of the desired-pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
The steady-state error is zero for a step input.	Open-loop gain has a pole at 1.	Open-loop gain $= D(z)\left(\frac{z+0.2}{(z-0.2)(z-1)}\right).$ which already has a pole at 1.
Maximum percent overshoot for a unit-step input	$M_p \approx 0.05 \pm 0.01.$	From the α-M_p curves, $\zeta = 0.7$ provides the broadest range of α values, where $-60^\circ < \alpha < -20^\circ$.
Settling time for a unit-step input	$\rho \le (0.02)^{1/(k_{2\%s}-1)}.$	For $t_{2\%s} = k_{2\%s}T \le 2.5\,\text{s}$, and $k_{2\%s} \le 2.5/0.5 = 5$, when $T = 0.5\,\text{s}$; $\rho \le (0.02)^{1/(5-1)} = 0.3761.$

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.15 \pm j0.3$.



$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T}e^{j\sqrt{1-\zeta^2}\omega T}$$

The deficiency angle, ϕ, needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

$$\phi + \measuredangle\big(z_d - (-0.2)\big) - \measuredangle\big(z_d - (0.2)\big) - \measuredangle\big(z_d - (1)\big) = (2k+1)\pi,$$

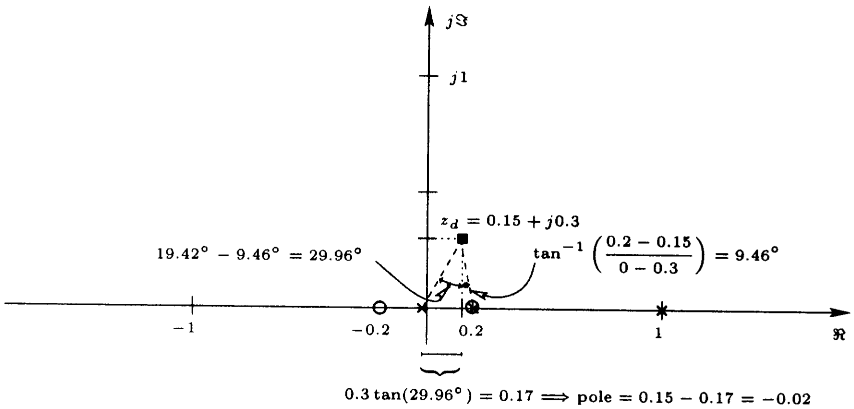
for an integer k. For $z_d = 0.15 + j0.3$,

$$\phi + \tan^{-1}\left(\frac{(0.3)-(0)}{(0.15)-(-0.2)}\right) - \tan^{-1}\left(\frac{(0.3)-(0)}{(0.15)-(0.2)}\right) - \tan^{-1}\left(\frac{(0.3)-(0)}{(0.15)-(1)}\right) = 180^\circ + k360^\circ,$$

$$\phi + 40.60^\circ - 99.46^\circ - 160.56^\circ = 180^\circ + k360^\circ,$$

or $\phi = 39.42^\circ$.

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at z_d. The best choice for cancelation is the pole at 0.2, since it is the slowest, and since cancelling the other pole at 1 prevents the satisfaction of the steady-state requirement.



From the above analysis,

$$D(z) = K\frac{z-0.2}{z+0.02}.$$

And the magnitude K is obtained from the magnitude condition at z_d.

$$\left|D(z)G(z)\right|_{z=z_d} = 1,$$

$$\left|K\frac{z+0.2}{(z+0.02)(z-1)}\right|_{z=0.15+j0.3} = 1,$$

or $K = 0.6743$. Therefore,

$$D(z) = 0.6743\frac{z-0.2}{z+0.02}$$

is one possible controller.

(b) Design another controller cascaded to the first one, such that the steady-state error for the unit-ramp input is reduced to one fourth of its value, and the location of the existing closed-loop poles stay approximately the same.

Solution: The system is type 1, so the steady-state error

$$e(\infty) = \frac{1}{K_v}.$$

Since $e_{\text{desired}}(\infty) = (1/4)e(\infty)$,

$$\frac{1}{K_{v_{\text{desired}}}} = \frac{1/4}{K_v},$$

or

$$K_{v_{\text{desired}}} = 4K_v.$$

To increase K_v to $K_{v_{\text{desired}}}$, we need to have a lag compensator with gain

$$\beta = 4.$$

For $\beta = 4$, the lag compensator becomes

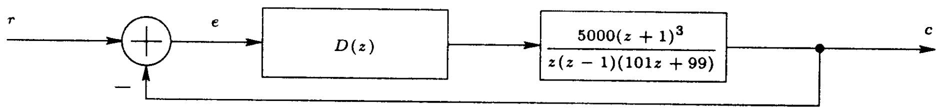
$$D'(z) = \frac{z-(1-1/T)}{z-(1-1/(\beta T))} = \frac{z-(1-1/T)}{z-(1-1/(4T))}.$$

To have the minimal effect on the existing poles, the pole and the zero of the compensator should be as close as possible to each other. We can accomplish this necessity by choosing the pole and the zero very close to one and a large value for T. For $T = 20$,

$$D'(z) = \frac{z - 0.95}{z - 0.9875}$$

is one possible compensator.

3. Consider the following feedback control system.



Design the simplest controller $D(z)$, such that the steady-state error for a unit-ramp input $e(\infty) \le 0.01$, and the gain margin of the system is about 20 dB. Assume the sampling period $T = 2\,\text{s}$.

Solution: For $G(z) = 5000(z+1)^3/\big(z(z-1)(101z+99)\big)$, the steady-state error for a unit-ramp input is given by

$$e(\infty) = \frac{1}{K_v},$$

where K_v is the velocity steady-state error coefficient; and

$$K_v = \lim_{z\to 1}\left(\left(\frac{z-1}{T}\right) D(z)G(z)\right) = \lim_{z\to 1}\left(\left(\frac{z-1}{T}\right) D(z)\left(\frac{5000(z+1)^3}{z(z-1)(101z+99)}\right)\right)$$

$$= \lim_{z\to 1}\left(D(z)\frac{5000(z+1)^3}{Tz(101z+99)}\right) = \left(D(1)\frac{5000(2)^3}{(2)(1)(200)}\right) = 100D(1).$$

Since $e_{\text{desired}}(\infty) \le 0.01$,

$$K_{v_{\text{desired}}} = \frac{1}{e_{\text{desired}}(\infty)} \ge 100,$$

and setting $K_v = K_{v_{\text{desired}}}$, we get $D(1) = 1$. The w-transform of G

$$\mathcal{W}\left[G\right](w) = \Big[G(z)\Big]_{z=\frac{1+(T/2)w}{1-(T/2)w}} = \left[\frac{5000(z+1)^3}{z(z-1)(101z+99)}\right]_{z=\frac{1+w}{1-w}}$$

$$= \frac{5000\left(\dfrac{1+w}{1-w}+1\right)^3}{\left(\dfrac{1+w}{1-w}\right)\left(\dfrac{1+w}{1-w}-1\right)\left(101\dfrac{1+w}{1-w}+99\right)} = \frac{5000\left(\dfrac{2}{1-w}\right)^3}{\left(\dfrac{1+w}{1-w}\right)\left(\dfrac{2w}{1-w}\right)\left(\dfrac{2(100+w)}{1-w}\right)}$$

$$= \frac{10000}{w(1+w)(100+w)} = \frac{100}{w(1+w)(1+w/100)}.$$

The phase-crossover frequency for $D(z) = 1$ is when

$$\measuredangle(\mathcal{W}\left[G\right](j\omega)) = -180°.$$

In our case,

$$\measuredangle\left(\frac{100}{j\omega_{\mathrm{p}}(1+j\omega_{\mathrm{p}})(1+j\omega_{\mathrm{p}}/100)}\right) = -180°,$$

$$\measuredangle(100) - \measuredangle(j\omega_{\mathrm{p}}) - \measuredangle(1+j\omega_{\mathrm{p}}) - \measuredangle(1+j\omega_{\mathrm{p}}/100) = -180°,$$

$$(0°) - (90°) - \tan^{-1}(\omega_{\mathrm{p}}) - \tan^{-1}(\omega_{\mathrm{p}}/100) = -180°,$$

$$\tan^{-1}(\omega_{\mathrm{p}}) + \tan^{-1}(\omega_{\mathrm{p}}/100) = 90°.$$

After taking the tangent of the above equation, we get

$$\frac{\left(\tan(\tan^{-1}(\omega_{\mathrm{p}}))\right)+\left(\tan(\tan^{-1}(\omega_{\mathrm{p}}/100))\right)}{1-\left(\tan(\tan^{-1}(\omega_{\mathrm{p}}))\right)\left(\tan(\tan^{-1}(\omega_{\mathrm{p}}/100))\right)} = \tan(90°),$$

$$\frac{(\omega_{\mathrm{p}}) + (\omega_{\mathrm{p}}/100)}{1-(\omega_{\mathrm{p}})(\omega_{\mathrm{p}}/100)} = \infty,$$

$$1 - \omega_{\mathrm{p}}^2/100 = 0,$$

or $\omega_{\mathrm{p}} = 10\,\mathrm{rad/s}$. Note that we could have easily determined the phase-crossover frequency from the Bode plots of $\mathcal{W}\left[G\right](j\omega)$ as well.

$$\text{Gain-Margin} = -\left|\mathcal{W}\left[G\right](j\omega_{\mathrm{p}})\right|_{\mathrm{dB}} = -20\log\left(\left|\mathcal{W}\left[G\right](j10)\right|\right) = 0.0864\,\mathrm{dB}.$$

We may proceed to design the desired compensator as a lead or a lag compensator. Here, we will consider both designs.

Lead compensator design

One approach to obtain a gain margin of 20 dB is to design for a new phase-crossover frequency, where the existing gain is −20 dB. To get the maximum phase-angle contribution, we choose the mid-frequency of the lead compensator at this frequency. In other words,

$$\left|\mathcal{W}\left[G\right](j\omega_{\mathrm{m}})\right|_{\mathrm{dB}} = -20\,\mathrm{dB},$$

or

$$\left|\frac{100}{j\omega_{\mathrm{m}}(1+j\omega_{\mathrm{m}})(1+j\omega_{\mathrm{m}}/100)}\right|_{\mathrm{dB}} = -20\,\mathrm{dB},$$

$$\left|\frac{100}{j\omega_{\mathrm{m}}(1+j\omega_{\mathrm{m}})(1+j\omega_{\mathrm{m}}/100)}\right| = 10^{-20/20} = 0.1.$$

Solving the above equation for ω_{m}, we get $\omega_{\mathrm{m}} \approx 31\,\mathrm{rad/s}$[1]. (Actually, $\omega_{\mathrm{m}} = 30.9018\,\mathrm{rad/s}$.) Therefore, if the new phase-crossover frequency is 31 rad/s, then the gain margin will be approximately 20 dB provided that the gain profile about the new phase-crossover frequency stays the same. The phase angle of a lead compensator needed at $\omega_{\mathrm{m}} = 31\,\mathrm{rad/s}$ is

$$\phi_{\mathrm{m}} \approx -180° - \measuredangle(\mathcal{W}\left[G\right](j\omega_{\mathrm{m}})) + 12° = -180° - (-195.38°) + 12° = 27.38°$$

[1]The mid-point of 10 rad/s and 100 rad/s is $\sqrt{(10)(100)}\,\mathrm{rad/s} = 31.6228\,\mathrm{rad/s}$.

assuming that 12° would compensate for the gain contribution of the lead compensator.

The parameter α of the lead compensator is given by

$$\alpha = \frac{1 - \sin(\phi_{\mathrm{m}})}{1 + \sin(\phi_{\mathrm{m}})} = 0.37,$$

and the lead compensator in w-transform domain

$$\mathcal{W}\left[D\right](w) = \frac{w/(\sqrt{\alpha}\omega_{\mathrm{m}}) + 1}{w/(\omega_{\mathrm{m}}/\sqrt{\alpha}) + 1} = \frac{w/18.86 + 1}{w/50.96 + 1}.$$

Finally, $D(z)$ is determined from

$$D(z) = \left[\mathcal{W}\left[D\right](w)\right]_{w=\frac{2}{T}\frac{z-1}{z+1}} = \left[\frac{w/18.86 + 1}{w/50.96 + 1}\right]_{w=\frac{z-1}{z+1}},$$

or

$$D(z) = 1.03\left(\frac{z + 0.90}{z + 0.96}\right).$$

Note here that the new phase-crossover frequency is moved due to the additional 12° added to compensate for the gain contribution. Indeed, with the designed lead compensator, the new phase-crossover frequency is 49.2 rad/s, and the gain margin is 22.56 dB.

Lag compensator design

Since the lag compensator needs to supply -20 dB at $\omega = 10$ rad/s without changing the phase-crossover frequency. the high corner-frequency (that is associated with the zero) of the compensator must be at least 10 times lower than the phase-crossover frequency ω_{p}. As a result,

$$\mathcal{W}\left[D\right](j\omega) = \frac{j\omega/(\omega_{\mathrm{p}}/50) + 1}{j\omega/(\omega_{\mathrm{p}}/(50\beta)) + 1},$$

where the frequency is 50 times lower. The gain requirement of the compensator sets the value of β, such that

$$-20\log(\beta) = -20 - 3,$$

where the -3 dB drop at the corner frequency is considered. In our case, we get $\beta \approx 14$. Therefore,

$$\mathcal{W}\left[D\right](w) = \frac{w/(10/50) + 1}{w/(10/(50(14))) + 1} = \frac{5w + 1}{70w + 1}.$$

Finally, $D(z)$ is determined from

$$D(z) = \left[\mathcal{W}\left[D\right](w)\right]_{w=\frac{2}{T}\frac{z-1}{z+1}} = \left[\frac{5w + 1}{70w + 1}\right]_{w=\frac{z-1}{z+1}},$$

or

$$D(z) = (6/71)\left(\frac{z - 2/3}{z - 69/71}\right) = 0.0845\left(\frac{z - 0.6667}{z - 0.9718}\right).$$

The new phase-crossover frequency is 9.0135 rad/s, and the gain margin is 21.20 dB.

Note that even though the lag-compensator design is considerably simpler, the bandwidth and the settling time of the system with the lag compensator are also considerable narrower and longer, respectively.