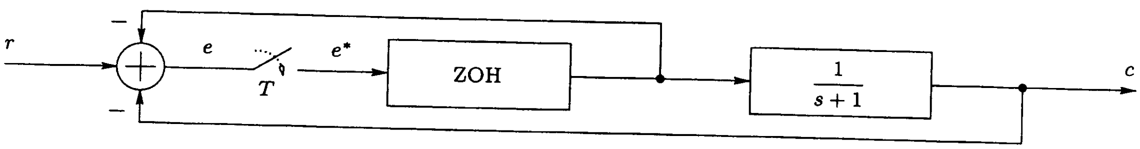
EE 331

Exam#1
75 minutes

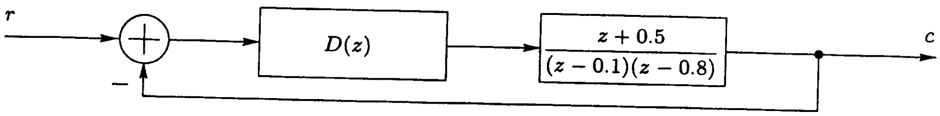
Mar. 05, 2002

1. Consider the following system with a sampling period of 0.1 second.



Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible. (30pts)

2. Consider the following feedback control system with a sampling period $T = 0.1\,\mathrm{s}$.



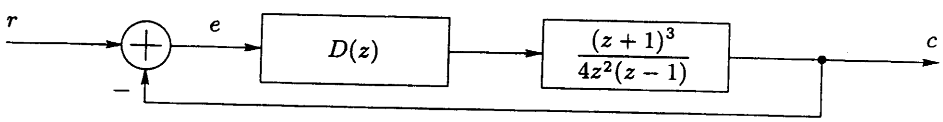
Design a first-order compensator $D(z)$, such that the following conditions are satisfied.

(a) The maximum percent overshoot is between 1% and 3% for the unit-step input.

(b) The 2% settling time is less than 0.4 second.

(35pts)

3. Consider the following feedback control system.



Design a first-order compensator, such that the steady-state error for a unit-ramp input $e(\infty) \leq (1/2)$, and the phase margin of the system is about 45°. Assume the sampling period $T = 2\,\mathrm{s}$. (35pts)

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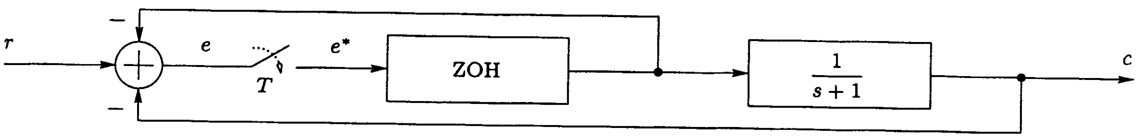
1. Consider the following system with a sampling period of 0.1 second.



Determine the transfer function $C(z)/R(z)$. Simplify the result as much as possible.

Solution: In order to be able to take the z transforms of signals, they need to be sampled or pseudo-sampled. Denoting the transfer function of the zero-order hold (ZOH) by G_{ZOH}, we have

$$\begin{aligned} E(s) &= R(s) - G_{\text{ZOH}}(s)E^*(s) - C(s) \\ &= R(s) - G_{\text{ZOH}}(s)E^*(s) - \frac{1}{s+1}G_{\text{ZOH}}(s)E^*(s). \end{aligned}$$

After the sampler, we have

$$E^*(s) = R^*(s) - G_{\text{ZOH}}(s)E^*(s) - \frac{1}{s+1}G_{\text{ZOH}}(s)E^*(s),$$

or

$$\left(1 + G_{\text{ZOH}}(s) + \frac{1}{s+1}G_{\text{ZOH}}(s)\right)E^*(s) = R^*(s).$$

Since all the signals are sampled, we may take the z transforms of the inverse Laplace transforms in the above equation.

$$\left(1 + \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[G_{\text{ZOH}}(s)\right]\right](z) + \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[(1/(s+1))\,G_{\text{ZOH}}(s)\right]\right](z)\right)E(z) = R(z).$$

To simplify the notation, we let

$$\begin{aligned} G_{\text{ZOH}}(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[G_{\text{ZOH}}(s)\right]\right](z), \\ (GG_{\text{ZOH}})(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[(1/(s+1))\,G_{\text{ZOH}}(s)\right]\right](z). \end{aligned}$$

Then,

$$E(z) = \frac{1}{1 + G_{\text{ZOH}}(z) + (GG_{\text{ZOH}})(z)}R(z).$$

We also have

$$C(s) = \frac{1}{s+1}G_{\text{ZOH}}(s)E^*(s),$$

and the z transform of the inverse Laplace transform on the pseudo-sampled output gives

$$\begin{aligned} C(z) &= \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[(1/(s+1))\,G_{\text{ZOH}}(s)\right]\right](z)E(z) \\ &= (GG_{\text{ZOH}})(z)E(z) \end{aligned}$$

Substituting the expression for $E(z)$ in the previous equation, we get

$$C(z) = \frac{(GG_{\mathrm{ZOH}})(z)}{1 + G_{\mathrm{ZOH}}(z) + (GG_{\mathrm{ZOH}})(z)} R(z).$$

Next, we need to determine the z-transform terms.

$$G_{\mathrm{ZOH}}(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{1}{s}\right]\right](z) = \left(\frac{z-1}{z}\right) \mathcal{Z}_k\left[\mathbf{1}(kT)\right](z)$$
$$= \left(\frac{z-1}{z}\right)\left(\frac{z}{z-1}\right) = 1.$$

$$(GG_{\mathrm{ZOH}})(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\left(\frac{1}{s}\right)\left(\frac{1}{s+1}\right)\right]\right](z)$$
$$= \left(\frac{z-1}{z}\right) \mathcal{Z}\left[\mathcal{L}_s^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right]\right](z) = \left(\frac{z-1}{z}\right)\left(\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right)$$
$$= 1 - \frac{z-1}{z-e^{-T}} = \frac{1-e^{-T}}{z-e^{-T}}.$$

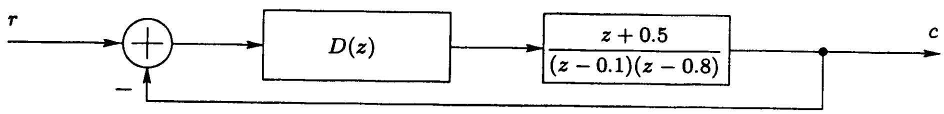
Substituting these expressions in the output expression, we get

$$\frac{C(z)}{R(z)} = \frac{((1-e^{-T})/(z-e^{-T}))}{1+1+((1-e^{-T})/(z-e^{-T}))}$$
$$= \frac{1-e^{-T}}{2z - 3e^{-T} + 1} = \frac{0.0952}{2z - 1.7145},$$

or

$$\frac{C(z)}{R(z)} = \frac{0.0476}{z - 0.8573}.$$

2. Consider the following feedback control system with a sampling period $T = 0.1$ s.



Design a first-order compensator $D(z)$, such that the following conditions are satisfied.

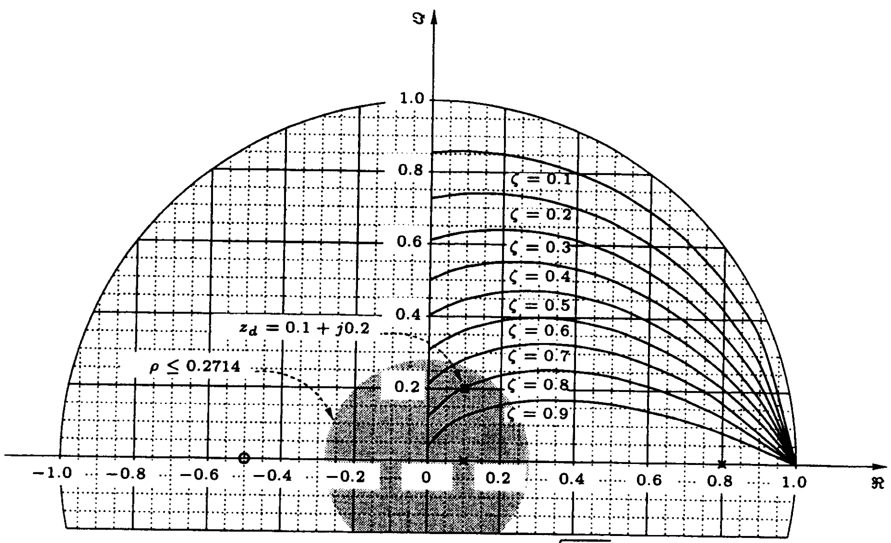
(a) The maximum percent overshoot is between 1% and 3% for the unit-step input.

(b) The 2% settling time is less than 0.4 second.

Solution: We determine the restrictions on the location of the desired-pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent overshoot for a unit-step input	$0.01 < M_p < 0.03.$	From the α-M_p curves, $\zeta = 0.8$ provides the broadest range of α values, where $-80^\circ < \alpha < 5^\circ$.
Settling time for a unit-step input	$\rho \leq (0.02)^{1/(k_{2\%s}-1)}$.	For $t_{2\%s} = k_{2\%s}T \leq 0.4\,\text{s}$, and $k_{2\%s} \leq 0.4/0.1 = 4$, when $T = 0.1\,\text{s}$; $\rho \leq (0.02)^{1/(4-1)} = 0.2714.$

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.1 \pm j0.2$.



$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T} e^{j\sqrt{1-\zeta^2}\omega T}$$

The deficiency angle, ϕ, needed at the desired location to ensure that one of the root-locus branches goes through the location, can be determined from the angular condition.

$$\phi + \measuredangle(z_d - (-0.5)) - \measuredangle(z_d - (0.1)) - \measuredangle(z_d - (0.8)) = (2k+1)\pi,$$

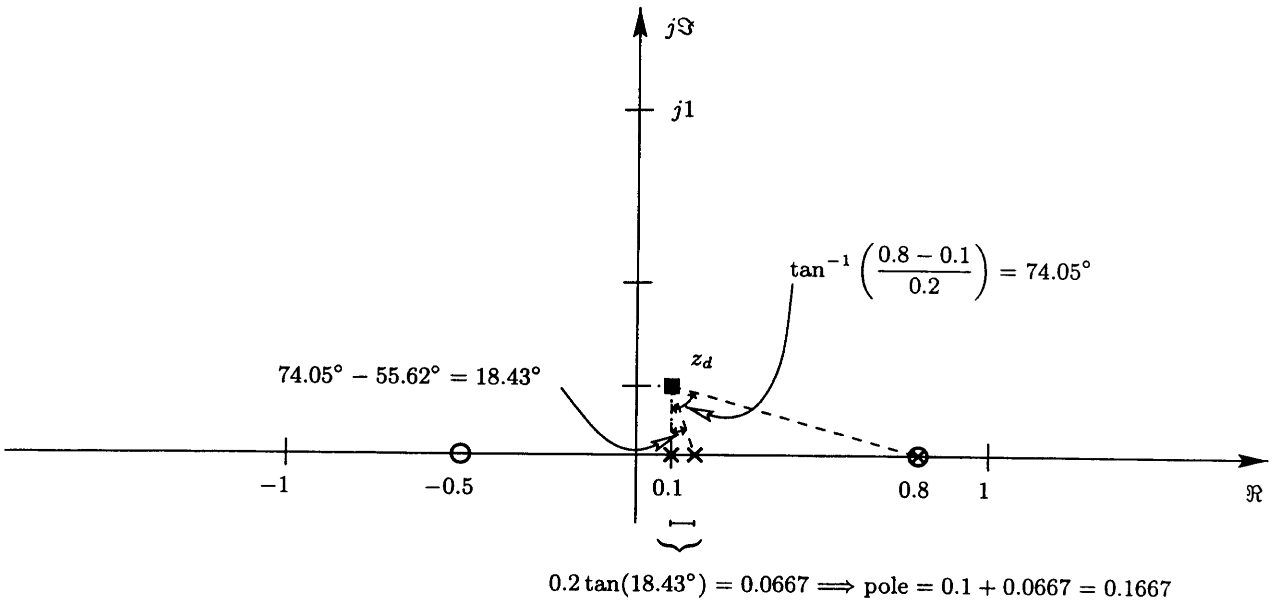
for an integer k. For $z_d = 0.1 + j0.2$,

$$\phi + \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(-0.5)}\right) - \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(0.1)}\right) - \tan^{-1}\left(\frac{(0.2)-(0)}{(0.1)-(0.8)}\right) = 180^\circ + k360^\circ.$$

$$\phi + 18.43° - 90° - 164.05° = 180° + k360°,$$

or $\phi = 55.62°$.

In order to preserve the system order so that transient specifications stay accurate, we need to cancel a pole or zero and place another one in such a way that the pole-zero combination provides the necessary deficiency angle at z_d. The best choice for cancelation is the pole at 0.8, since it is the slowest.



From the above analysis,

$$D(z) = K\frac{z-0.8}{z-0.1667}.$$

And the magnitude K is obtained from the magnitude condition at z_d.

$$\left|D(z)G(z)\right|_{z=z_d} = 1,$$

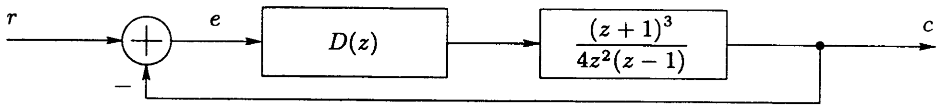
$$\left|K\frac{z+0.5}{(z-0.1667)(z-0.1)}\right|_{z=0.1+j0.2} = 1,$$

or $K = 0.0667$. Therefore,

$$D(z) = 0.0667\frac{z-0.8}{z-0.1667} = (1/15)\frac{z-(4/5)}{z-(5/30)}$$

is one possible controller.

3. Consider the following feedback control system.



Design a first-order compensator, such that the steady-state error for a unit-ramp input $e(\infty) \le (1/2)$, and the phase margin of the system is about 45°. Assume the sampling period $T = 2\,\mathrm{s}$.

Solution: For $G(z) = (z+1)^3/(4z^2(z-1))$, the steady-state error for a unit-ramp input is given by

$$e(\infty) = \frac{1}{K_v},$$

where K_v is the velocity steady-state error coefficient; and

$$K_v = \lim_{z\to 1}\left(\left(\frac{z-1}{T}\right)D(z)G(z)\right) = \lim_{z\to 1}\left(\left(\frac{z-1}{T}\right)D(z)\left(\frac{(z+1)^3}{4z^2(z-1)}\right)\right)$$
$$= \lim_{z\to 1}\left(D(z)\frac{(z+1)^3}{4Tz^2}\right) = \left(D(1)\frac{(2)^3}{4(2)(1)^2}\right) = D(1).$$

Since $e_{\text{desired}}(\infty) \le (1/2)$,

$$K_{v_{\text{desired}}} = \frac{1}{e_{\text{desired}}(\infty)} \ge 2,$$

and setting $K_v = K_{v_{\text{desired}}}$, we get $D(1) = 2$. Let $D(z) = 2D'(z)$, and $G'(z) = 2G(z)$. The w-transform of G'

$$\mathcal{W}\left[G'\right](w) = 2\Big[G(z)\Big]_{z=\frac{1+(T/2)w}{1-(T/2)w}} = \left[\frac{2(z+1)^3}{4z^2(z-1)}\right]_{z=\frac{1+w}{1-w}}$$

$$= \frac{2\left(\dfrac{1+w}{1-w}+1\right)^3}{4\left(\dfrac{1+w}{1-w}\right)^2\left(\dfrac{1+w}{1-w}-1\right)} = \frac{2\left(\dfrac{2}{1-w}\right)^3}{4\left(\dfrac{1+w}{1-w}\right)^2\left(\dfrac{2w}{1-w}\right)} = \frac{2}{w(1+w)^2}.$$

We may proceed to design the desired compensator as a lead or a lag compensator. Here, we will consider both designs.

Lead compensator design

The gain-crossover frequency for $D'(z) = 1$ or $\mathcal{W}\left[D'\right](w) = 0$ is when

$$\left|\mathcal{W}\left[G'\right](j\omega)\right|_{\text{dB}} = 0,$$

or

$$\left|\mathcal{W}\left[G'\right](j\omega)\right| = 1.$$

In our case,

$$\left|\frac{2}{j\omega_g(1+j\omega_g)^2}\right| = 1,$$

or $\omega_g = 1\,\text{rad/s}$.

$$\text{Phase-Margin} = \measuredangle(\mathcal{W}\left[G'\right](j\omega_g))_{\omega_g=1} + 180^\circ = -180^\circ + 180^\circ = 0^\circ.$$

Since Phase-Margin$_{\text{desired}}$ = 45°, the phase angle of a lead compensator needed at the mid-frequency

$$\phi_m \approx \text{Phase-Margin}_{\text{desired}} - \text{Phase-Margin} + 12^\circ = 45^\circ - 0^\circ + 12^\circ = 57^\circ$$

assuming that 12° would compensate for the displaced gain-crossover frequency due to the cascaded lead compensator.

The parameter α of the lead compensator is given by

$$\alpha = \frac{1-\sin(\phi_m)}{1+\sin(\phi_m)} = 0.0877.$$

The mid-frequency ω_m can be determined from

$$\left|\mathcal{W}\left[G'\right](j\omega)\right|_{\text{dB}} = -20\log(1/\sqrt{\alpha}),$$

or

$$\left|\mathcal{W}\left[G'\right](j\omega)\right| = \sqrt{\alpha}.$$

In our case,

$$\left|\frac{2}{j\omega_m(1+j\omega_m)^2}\right| = \sqrt{0.0877},$$

or $\omega_m = 1.7143\,\text{rad/s}$; and the lead compensator in w-transform domain

$$\mathcal{W}\left[D'\right](j\omega) = \frac{j\omega/(\sqrt{\alpha}\omega_m)+1}{j\omega/(\omega_m/\alpha)+1} = \frac{j\omega/0.5078+1}{j\omega/5.7872+1}.$$

Finally, $D'(z)$ is determined from

$$D'(z) = \left[\mathcal{W}\left[D'\right](w)\right]_{w=\frac{2}{T}\frac{z-1}{z+1}} = 2.5318\left(\frac{z-0.3264}{z+0.7053}\right).$$

Since $D(z) = 2D'(z)$, we have

$$D(z) = 5.0636\left(\frac{z-0.3264}{z+0.7053}\right).$$

Lag compensator design

The desired gain-crossover frequency $\omega_{g_{\text{desired}}}$ is when the phase angle is

$$\measuredangle(\mathcal{W}\left[G'\right](j\omega_{g_{\text{desired}}})) = -180^\circ + \text{Phase-Margin}_{\text{desired}} + 12^\circ = -180^\circ + 45^\circ + 12^\circ = -123^\circ$$

assuming that 12° would compensate for the added negative phase angle due to the cascaded lag compensator. Since

$$\measuredangle(\mathcal{W}\left[G'\right](j\omega_{g_{\text{desired}}})) = -\measuredangle(j\omega_{g_{\text{desired}}}) - 2\measuredangle(1+j\omega_{g_{\text{desired}}}) = 90^\circ - 2\tan^{-1}(\omega_{g_{\text{desired}}}) = -123^\circ,$$

we get $\tan^{-1}(\omega_{g_{\text{desired}}}) = 33°$, or $\omega_{g_{\text{desired}}} = 0.2962\,\text{rad/s}$. The gain at this frequency

$$\left|\mathcal{W}\left[G'\right](j\omega)\right|_{[\omega=0.2962]_{\text{dB}}} = 20\log\left|\frac{2}{j\omega(1+j\omega)^2}\right|_{\omega=0.2962} = 20\log(6.2072).$$

Since this gain is to be reduced to 0 dB by the lag compensator, the compensator gain β is such that

$$|\beta|_{\text{dB}} = 20\log(\beta) = 20\log(6.2072),$$

or $\beta \approx 6.2$.

Choosing the cut-off frequency of the compensator 10 times slower than the desired gain-crossover frequency, so that the negative phase angle of the compensator doesn't effect the phase angle directly, we get the lag compensator in w-transform domain as

$$\mathcal{W}\left[D'\right](j\omega) = \frac{j\omega/(\omega_{g_{\text{desired}}}/10)+1}{j\omega/(\omega_{g_{\text{desired}}}/(10\beta))+1} = \frac{j\omega/0.02962+1}{j\omega/0.00478+1}.$$

Finally, $D'(z)$ is determined from

$$D'(z) = \left[\mathcal{W}\left[D'\right](w)\right]_{w=\frac{2}{T}\frac{z-1}{z+1}} = 0.1653\left(\frac{z-0.9425}{z-0.9905}\right).$$

Since $D(z) = 2D'(z)$, we have

$$D(z) = 0.3306\left(\frac{z-0.9425}{z-0.9905}\right).$$