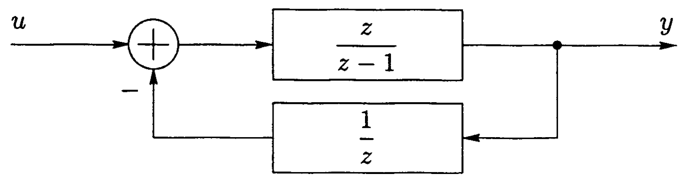
EE 331 **Exam#2** May 02, 2002

75 minutes

1. The block diagram of a control system is given in the following figure.
   (a) Obtain a state-space representation of the system without any block-diagram reduction. (15pts)
   (b) Determine the state variable $\mathbf{x}(k)$ for all $k \geq 0$; when $y(-1) = 0$, $y(0) = 1$, and $u(k) = (-1)^k \mathbf{1}(k)$. (20pts)



2. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix} \mathbf{u}(k),$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{u}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

   (a) Determine whether or not all any final state can be achieved from any other state by selecting a control sequence. If such a selection is possible, then determine the control sequence that would achieve the state $\begin{bmatrix} -5 & 2 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$. (15pts)
   (b) Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible, then determine the initial condition $\mathbf{x}(0)$, when the output sequence is $\{\, y(k) \mid k = 0, 1, 2, \dots \} = \{\, 2,\ 3.6,\ 4.72,\ 7.144, \dots \}$ for the input sequence $\{\, \mathbf{u}(k) \mid k = 0, 1, 2, \dots \} = \{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & -1 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \dots \}$. (15pts)

3. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix} \mathbf{u}(k),$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

   (a) Design a state-feedback controller, such that the poles of the closed-loop system are at $z = 0.1$ and $z = 0.8$. (20pts)
   (b) Implement the controller in the previous part by assuming that only the output is available. (15pts)

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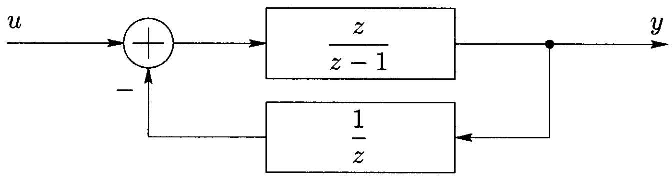
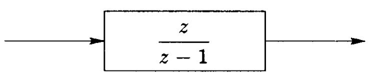
Solutions

1. The block diagram of a control system is given in the following figure.

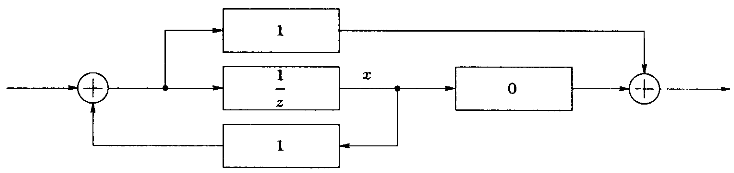


(a) Obtain a state-space representation of the system without any block-diagram reduction.

Solution: In order to obtain a state-space representation without any block-diagram reduction or without determining the closed-loop transfer function, we need to realize the individual blocks and use the complete block diagram to generate the state-space equations.

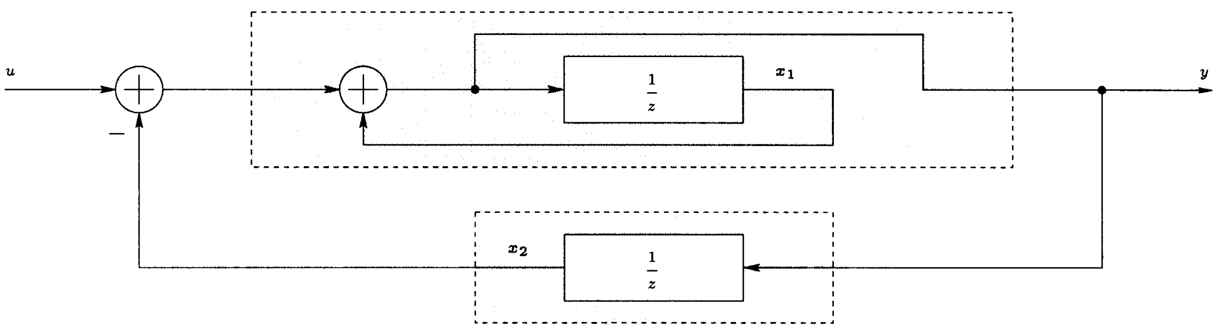


(a) The feedforward gain block.



(b) Controller realization form.

There is no need to generate a realization for the feedback gain block, since it is already in a realization form. The connected and "expanded" block diagram is shown below.



After assigning the state variables as shown in the figure, we obtain

$$x_1(k+1) = x_1(k) + \big(u(k) - x_2(k)\big) = x_1(k) - x_2(k) + u(k),$$
$$x_2(k+1) = x_1(k+1) = x_1(k) - x_2(k) + u(k),$$

and

$$y(k) = x_1(k+1) = x_1(k) - x_2(k) + u(k).$$

And the state-space representation is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k),$$
$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + [\,1\,]u(k).$$

Note here that the observer realization form results in the same realization diagram.

(b) Determine the state variable $\mathbf{x}(k)$ for all integer $k \geq 0$; when $y(-1) = 0$, $y(0) = 1$, and $u(k) = (-1)^k \mathbf{1}(k)$.

Solution: The state variable solution, which is obtain by repeated application of the state equation, is given by

$$\begin{aligned}\mathbf{x}(k) &= A^k \mathbf{x}(0) + A^{k-1} B\mathbf{u}(0) + A^{k-2} B\mathbf{u}(1) + \ldots + AB\mathbf{u}(k-2) + B\mathbf{u}(k-1) \\ &= A^k \mathbf{x}(0) + \sum_{i=1}^{k} A^{k-i} B\mathbf{u}(i-1),\end{aligned}$$

for $k \geq 0$, and for a discrete-time system described by

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$
$$\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For our system, the input variable $u(k)$ for $k \geq 0$ is given, but we need to determine $\mathbf{x}(0)$ and A^k.

The initial condition $\mathbf{x}(0)$ may be determined from the output values provided. From the state-space equations, we have

$$y(0) = x_1(0) - x_2(0) + u(0) = 1,$$

and

$$y(-1) = x_1(-1) - x_2(-1) + u(-1) = 0.$$

On the other hand, we also have

$$x_1(0) = x_1(-1) - x_2(-1) + u(-1),$$

and

$$x_2(0) = x_1(-1) - x_2(-1) + u(-1).$$

From the expression of $y(-1)$, we observe that $x_1(0) = y(-1) = 0$ and $x_2(0) = y(-1) = 0$; or

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Even though $\mathbf{x}$ is the zero vector, we still need to determine A^k for the other expressions in the state-variable solution.

We may use a number of different methods to determine A^k. One of these methods is by repeated multiplications of A, until we observe a pattern. Although, this method is quite simple to proceed, we have to remember that sometimes the pattern may not be so obvious. Another method is from the z-transform of the state-space equations, where we get

$$A^{k-1} = \mathcal{Z}^{-1}\left[(zI - A)^{-1}\right],$$

or

$$A^k = \mathcal{Z}^{-1}\left[z(zI - A)^{-1}\right].$$

Here, we will use both of these methods to demonstrate the use of each method. Since

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix},$$

we get

$$A^2 = AA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A^3 = A^2 A = 0$$

$$A^4 = A^3 A = 0,$$

or $A^k = 0$ for integer $k \geq 2$. Under these values of $\mathbf{x}(0)$ and A^k, we get

$$\begin{aligned}
\mathbf{x}(k) &= A^k\mathbf{x}(0) + \sum_{i=1}^{k} A^{k-i}B\mathbf{u}(i-1) \\
&= ABu(k-2) + Bu(k-1) \\
&= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(-1)^{k-2}\mathbb{1}(k-2) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(-1)^{k-1}\mathbb{1}(k-1) \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}(-1)^{k-2}\mathbb{1}(k-2) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(-1)^{k-1}\mathbb{1}(k-1) \\
&= \begin{bmatrix} (-1)^{k-1} \\ (-1)^{k-1} \end{bmatrix}\mathbb{1}(k-1).
\end{aligned}$$

Therefore,

$$\mathbf{x}(k) = \begin{cases} \begin{bmatrix} 0 & 0 \end{bmatrix}^T, & \text{if } k \leq 0; \\ \begin{bmatrix} (-1)^{k-1} & (-1)^{k-1} \end{bmatrix}^T, & \text{if } k \geq 1. \end{cases}$$

2. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix} \mathbf{u}(k),$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{u}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

(a) Determine whether or not all any final state can be achieved from any other state by selecting a control sequence. If such a selection is possible, then determine the control sequence that would achieve the state $\begin{bmatrix} -5 & 2 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

Solution: The property of being able to reach to any arbitrary final state from any other initial state is called reachability. One method to check the reachability of the system is by checking the rank of the controllability matrix

$$\mathcal{C}(A, B) = \left[\ B \mid AB \mid \cdots \mid A^{n-1}B\ \right].$$

In our case, the system order $n = 2$, and

$$\mathcal{C}(A, B) = \left[\ B \mid AB\ \right] = \left[\begin{array}{cc|cc} -9 & 4 & -9 & 0.8 \\ 3 & -1 & 3 & -0.2 \end{array}\right].$$

In our system, we need to have 2 linearly independent rows or columns of $\mathcal{C}$ for reachability. Since the first two columns of $\mathcal{C}$ are linearly independent, we conclude that the system is reachable and any final state can be achieved from any other state by selecting a control sequence. To determine the control sequence that would achieve the state $\begin{bmatrix} -5 & 2 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$, we may use a formula that can be derived by repeatedly applying the state-space equation.

$$\mathbf{x}(n) - A^n \mathbf{x}(0) = \mathcal{C}(A, B) \begin{bmatrix} \mathbf{u}(n-1) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}$$

for an nth order discrete-time system described by

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$

$$\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. In our case, $n = 2$, and

$$\begin{bmatrix} -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left[\begin{array}{cc|cc} -9 & 4 & -9 & 0.8 \\ 3 & -1 & 3 & -0.2 \end{array}\right] \left[\begin{array}{c} u_1(1) \\ u_2(1) \\ \hline u_1(0) \\ u_1(0) \end{array}\right].$$

In order to solve for the control variable, we may use the pseudo-inverse of $\mathcal{C}(A,B)$, such that $\mathcal{C}(A,B)^{\#} = \mathcal{C}(A,B)\big(\mathcal{C}(A,B)\mathcal{C}(A,B)^T\big)^{-1}$, that exists when $\mathcal{C}(A,B)$ has full rank. As a result,

$$\begin{bmatrix} u_1(1) \\ u_2(1) \\ \hline u_1(0) \\ u_1(0) \end{bmatrix} = \begin{bmatrix} -9 & 3 \\ 4 & -1 \\ -9 & 3 \\ 0.8 & -0.2 \end{bmatrix} \left(\begin{bmatrix} -9 & 4 & -9 & 0.8 \\ 3 & -1 & 3 & -0.2 \end{bmatrix} \begin{bmatrix} -9 & 3 \\ 4 & -1 \\ -9 & 3 \\ 0.8 & -0.2 \end{bmatrix} \right)^{-1} \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.50 \\ 0.96 \\ \hline 0.50 \\ 0.19 \end{bmatrix}.$$

Therefore, the control sequence, that would achieve the desired behavior in 2 steps, is

$$\mathbf{u}(0) \approx \begin{bmatrix} 0.50 \\ 0.19 \end{bmatrix}, \text{ and } \mathbf{u}(1) \approx \begin{bmatrix} 0.50 \\ 0.96 \end{bmatrix}.$$

However in this case, the controllability index of the system is 1, which is observed from the minimum number of the $A^{k-1}B$ terms that needs to be included in $\mathcal{C}(A,B)$ to reach full rank. When we consider the $\mathcal{C}(A,B)$ matrix, we observe that the first term B generates the rank of the $\mathcal{C}(A,B)$ matrix, and the next term AB is not necessary for the rank requirement. As a result, we should be able to achieve the final state in 1 step.

$$\mathbf{x}(1) - A^1\mathbf{x}(0) = \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} \mathbf{u}(0) \end{bmatrix}.$$

Since B is invertible, solving the above equation for $\mathbf{u}(0)$, we get

$$\mathbf{u}(0) = \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix}^{-1} \left(\begin{bmatrix} -5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right),$$

or the control sequence, that would achieve the desired behavior in 1 step, is

$$\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(b) Determine whether or not an initial condition can be uniquely determined by observing the future values of the output and control. If such an observation is possible, then determine the initial condition $\mathbf{x}(0)$, when the output sequence is $\{\, y(k) \mid k = 0,1,2,\dots \} = \{\, 2,\ 3.6,\ 4.72,\ 7.144, \dots \}$ for the input sequence $\{\, \mathbf{u}(k) \mid k = 0,1,2,\dots \} = \{\, [\, 1 \;\; 1\,]^T, [\, -1 \;\; -1\,]^T, [\, 1 \;\; -1\,]^T, [\, 0 \;\; 0\,]^T, \dots \}$.

Solution: The property of determining the initial condition from the future values of the output is the observability property. To ensure observability of the system, the rank of the observability matrix should be full. The observability matrix for an nth order system

$$\mathcal{O}(C,A) = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix},$$

where A and C are the state and the output matrices of the system, respectively. In our case,

$$\mathcal{O}(C,A) = \left[\begin{array}{c} C \\ \hline CA \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ \hline 0.8 & 3.4 \end{array}\right].$$

In our system, we need to have 2 linearly independent rows or columns of $\mathcal{O}$ for reachability. Since the determinant of $\mathcal{O}$ is non-zero, we conclude that the system is observable and the initial state variable can be obtained from the output variables. To determine the initial conditions, we may use a formula that can be derived by repeatedly applying the output equation.

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(n-1) \end{bmatrix} = \mathcal{O}(C,A)\mathbf{x}(0) + \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ C^{n-1}B & C^{n-2}B & \cdots & D \end{bmatrix} \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \mathbf{u}(n-1) \end{bmatrix}.$$

In our case, $n = 2$, and

$$\begin{bmatrix} 2 \\ 3.6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.8 & 3.4 \end{bmatrix} \mathbf{x}(0) + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}.$$

Solving for the unknown initial condition, we get

$$\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \\ 0.8 & 3.4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 2 \\ 3.6 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \right);$$

and after some matrix manipulations, we obtain

$$\mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

3. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix} \mathbf{u}(k),$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k),$$

where $\mathbf{u}$, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

(a) Design a state-feedback controller, such that the poles of the closed-loop system are at $z = 0.1$ and $z = 0.8$.

Solution: Based on the desired-pole locations, the desired characteristic polynomial is given by

$$q_{c_d}(z) = \big(z - (0.1)\big)\big(z - (0.8)\big) = z^2 - 0.9z + 0.08.$$

Assuming

$$u(k) = -K\mathbf{x}(k) = -\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}\mathbf{x}(k)$$

for some state-feedback matrix K. The characteristic polynomial of the system under state-feedback control can be determined from the denominator of the transfer function, such that

$$\begin{aligned} q_c(z) &= \det\big(zI - (A - BK)\big) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} - \begin{bmatrix} -9 & 4 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}\right)\right) \\ &= z^2 + (-1.2 - 9k_1 + 3k_2 + 4k_3 - k_4)z + (0.2 + 1.8k_1 - 0.6k_2 - 4k_3 + k_4 - 3k_1k_4 + 3k_2k_3). \end{aligned}$$

Setting $q_c(z) = q_{c_d}(z)$, we get

$$0.2 + 1.8k_1 - 0.6k_2 - 4k_3 + k_4 - 3k_1k_4 + 3k_2k_3 = 0.08,$$

and

$$-1.2 - 9k_1 + 3k_2 + 4k_3 - k_4 = -0.9.$$

Since we have only 2 equations and 4 unknowns, we have some freedom of choosing two of the unknowns arbitrarily. One possible choice would be $k_3 = -1$ and $k_4 = 1$. For this choice, we get $k_1 = -0.1033$, and $k_2 = 1.4567$. As a result, we have

$$K = \begin{bmatrix} -0.1033 & 1.4567 \\ -1 & 1 \end{bmatrix},$$

or

$$u(k) = \begin{bmatrix} 0.1033 & -1.4567 \\ 1 & -1 \end{bmatrix}\mathbf{x}(k).$$

If we would like to determine a more general solution, we may let $k_3 = a$, and $k_4 = b$. In this general case, we have

$$k_1 = \frac{-0.18 + 8.7a + 12a^2 - 2.4b - 3ab}{27a - 9b},$$

$$k_2 = \frac{-0.54 + 28.8a - 8.1b + 12ab - 3b^2}{27a - 9b},$$

as long as $27a - 9b \neq 0$. Therefore, the general form of the state-feedback control is

$$u(k) = \begin{bmatrix} \left(\dfrac{0.18 - 8.7a - 12a^2 + 2.4b + 3ab}{27a - 9b}\right) & \left(\dfrac{0.54 - 28.8a + 8.1b - 12ab + 3b^2}{27a - 9b}\right) \\ -a & -b \end{bmatrix}\mathbf{x}(k),$$

for any a and b such that $b \neq 3a$.

(b) Implement the controller in the previous part by assuming that only the output is available.

Solution: When only the output is available, state-feedback control can still be implemented if an observer is used. If a system is reachable, we can place the closed-loop poles of the observer at any desired location via error-feedback control. So assume

$$\mathbf{e}(k) = L\big(y(k) - \hat{y}(k)\big) = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \big(y(k) - \hat{y}(k)\big)$$

for some observer-error gain matrix L, where $\hat{y}$ is the observer output variable. Arbitrarily assuming that the observer poles are at 0.01 and 0.01, the desired observer-characteristic polynomial

$$q_{\mathrm{o_d}}(z) = (z - 0.01)(z - 0.01) = z^2 - 0.02z + 0.0001.$$

The observer-characteristic polynomial q_{o} under the error-feedback control can be determined from the denominator of the transfer function of the observer, such that

$$\begin{aligned} q_{\mathrm{o}}(z) &= \det\big(zI - (A - LC)\big) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} -2.2 & -9.6 \\ 0.8 & 3.4 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right)\right) \\ &= z^2 + (-1.2 + l_1)z + (0.2 - 3.4l_1 - 9.6l_2). \end{aligned}$$

Setting $q_{\mathrm{o}}(z) = q_{\mathrm{o_d}}(z)$, we get

$$0.2 - 3.4l_1 - 9.6l_2 = 0.0001,$$

and

$$-1.2 + l_1 = -0.02.$$

These two equations give $l_1 = 1.18$ and $l_2 = -0.3971$, and we obtain

$$\mathbf{e}(k) = \begin{bmatrix} 1.1800 \\ -0.3971 \end{bmatrix} \big(y(k) - \hat{y}(k)\big),$$

where $\mathbf{e}$ and $\hat{y}$ are the error-feedback control and the observer output variables, respectively.