EE 331 **Exam#2** **May 06, 2003**

75 minutes

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1. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 1/2 & 1 \\ 0 & -1/2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$

where u and $\mathbf{x}$ are the input and the state variables, respectively. Determine the solution $\mathbf{x}(k)$ for all $k \geq 0$, when $u(k) = \mathbb{1}(k)$. Simplify the expression as much as possible. (25pts)

2. A discrete-time control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(k),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

(a) Specify the minimum length control sequence necessary to transfer the state of this system from $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. (15pts)

(b) Specify the control necessary to maintain the system in that state. Briefly discuss your result. (10pts)

3. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.08 & 0.9 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \mathbf{x}(k),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively, and a sampling period $T = 0.1\,\text{s}$.

(a) Design a state-feedback controller, such that the following conditions are satisfied.

i. The maximum percent overshoot is between 1% and 3% for a unit-step input.

ii. The 2% settling time is less than 0.4 second.

(20pts)

(b) Design the necessary additions to the controller assuming that only the output is available. (10pts)

4. Consider a system described by the difference equation

$$x(k+1) = -x(k) + u(k),$$

where x and u are the state and the input variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = \frac{1}{2}\sum_{k=0}^{2}\left(x^2(k) + 4u^2(k)\right),$$

when $x(0) = -1$ and $x(3) = 0$. (20pts)

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1. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 1/2 & 1 \\ 0 & -1/2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k),$$

where u and $\mathbf{x}$ are the input and the state variables, respectively. Determine the solution $\mathbf{x}(k)$ for all $k \geq 0$, when $u(k) = \mathbb{1}(k)$. Simplify the expression as much as possible.

Solution: The solution to the discrete-time linear control system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),$$

where $\mathbf{u}$ and $\mathbf{x}$ are the input and the state variables, respectively, is given by

$$\mathbf{x}(k) = A^k\mathbf{x}(0) + \sum_{i=1}^{k} A^{k-i}B\mathbf{u}(i-1).$$

In this expression, A^k can be obtained in various methods. The method, that is based on the z-transform of the state-space equations, gives

$$A^k = \mathcal{Z}^{-1}\left[z(zI-A)^{-1}\right].$$

In our case,

$$A^k = \mathcal{Z}_z^{-1}\left[z(zI-A)^{-1}\right](k) = \mathcal{Z}_z^{-1}\left[z\begin{bmatrix} z-1/2 & -1 \\ 0 & z+1/2 \end{bmatrix}^{-1}\right](k)$$

$$= \mathcal{Z}_z^{-1}\left[z\begin{bmatrix} \dfrac{z+1/2}{(z+1/2)(z-1/2)} & \dfrac{1}{(z+1/2)(z-1/2)} \\ 0 & \dfrac{z-1/2}{(z+1/2)(z-1/2)} \end{bmatrix}\right](k)$$

$$= \mathcal{Z}_z^{-1}\begin{bmatrix} \dfrac{z}{z-1/2} & \dfrac{z}{(z+1/2)(z-1/2)} \\ 0 & \dfrac{z}{z+1/2} \end{bmatrix}(k)$$

$$= \mathcal{Z}_z^{-1}\begin{bmatrix} \dfrac{z}{z-1/2} & \dfrac{z}{z-1/2} - \dfrac{z}{z+1/2} \\ 0 & \dfrac{z}{z+1/2} \end{bmatrix}(k)$$

$$= \begin{bmatrix} (1/2)^k & (1/2)^k - (-1/2)^k \\ 0 & (-1/2)^k \end{bmatrix}$$

for $k \geq 0$. Therefore, we get

$$
\begin{aligned}
\mathbf{x}(k) &= A^k\mathbf{x}(0) + \sum_{i=1}^{k} A^{k-i}B\mathbf{u}(i-1) \\
&= \begin{bmatrix} (1/2)^k & (1/2)^k - (-1/2)^k \\ 0 & (-1/2)^k \end{bmatrix}\mathbf{x}(0) \\
&\quad + \sum_{i=1}^{k}\begin{bmatrix} (1/2)^{k-i} & (1/2)^{k-i} - (-1/2)^{k-i} \\ 0 & (-1/2)^{k-i} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} u(i-1) \\
&= \begin{bmatrix} (1/2)^k & (1/2)^k - (-1/2)^k \\ 0 & (-1/2)^k \end{bmatrix}\mathbf{x}(0) + \sum_{i=1}^{k}\begin{bmatrix} (1/2)^{k-i} \\ 0 \end{bmatrix} u(i-1).
\end{aligned}
$$

For $u(k) = \mathbb{1}(k)$, and $\sum_{i=1}^{k}(1/2)^{k-i} = \sum_{j=0}^{k-1}(1/2)^j = \left(1-(1/2)^k\right)/\left(1-(1/2)\right) = 2\left(1-(1/2)^k\right)$; we get

$$
\mathbf{x}(k) = \begin{bmatrix} (1/2)^k & (1/2)^k - (-1/2)^k \\ 0 & (-1/2)^k \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} 2 - 2(1/2)^k \\ 0 \end{bmatrix} \quad \text{for } k \geq 0.
$$

2. A discrete-time control system is described by

$$
\begin{aligned}
\mathbf{x}(k+1) &= \begin{bmatrix} 0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(k), \\
y(k) &= \begin{bmatrix} 1 & 1 \end{bmatrix}\mathbf{x}(k),
\end{aligned}
$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively.

(a) Specify the minimum length control sequence necessary to transfer the state of this system from $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

Solution: To determine the control sequence that would achieve the state $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ from the initial state $\begin{bmatrix} 0 & 0 \end{bmatrix}^T$, we may use a formula that can be derived by repeatedly applying the state-space equation.

$$
\mathbf{x}(k) - A^k\mathbf{x}(0) = \begin{bmatrix} B \mid AB \mid \cdots \mid A^{k-1}B \end{bmatrix}\begin{bmatrix} \mathbf{u}(k-1) \\ \vdots \\ \mathbf{u}(0) \end{bmatrix}
$$

for an nth order discrete-time system described by

$$
\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k), \\
\mathbf{y}(k) &= C\mathbf{x}(k) + D\mathbf{u}(k),
\end{aligned}
$$

where $\mathbf{u}$, $\mathbf{x}$, and $\mathbf{y}$ are the input, the state, and the output variables, respectively. For the minimum length control sequence, first we choose $k = 1$;

$$
\mathbf{x}(1) - A\mathbf{x}(0) = Bu(0),
$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(0).$$

The above matrix equation has two individual equations, the first one gives $u(0) = 2$, whereas the second one gives $u(0) = 4$. Since there is no single value of $u(0)$ that can satisfy both of the equations, we conclude that there is no solution for the above matrix equation or the final state may not be achieved in one step.

Next, we choose $k = 2$;

$$\mathbf{x}(2) - A^2\mathbf{x}(0) = \left[\, B \mid AB \,\right] \begin{bmatrix} \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \left[\begin{array}{c|c} 0.500 & 0.400 \\ 0.250 & 0.025 \end{array}\right] \begin{bmatrix} \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix}.$$

Solving the above matrix equation, we get

$$\begin{bmatrix} \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix} = \begin{bmatrix} 0.500 & 0.400 \\ 0.250 & 0.025 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\frac{1}{-0.0875}\right) \begin{bmatrix} 0.025 & -0.400 \\ -0.250 & 0.500 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \approx \begin{bmatrix} 4.286 \\ -2.857 \end{bmatrix}.$$

Note here that for $k = 2 = n$, we get the controllability matrix in the above equation; and since the controllability matrix is invertible, the system is reachable. If we couldn't have found a solution for the above matrix equation, then there would have been no need to try for $k > 2 = n$.

In our case, the minimum length control sequence, that would achieve the desired behavior, is

$$\{\, u(0),\, u(1) \,\} \approx \{\, -2.857,\, 4.286 \,\}.$$

(b) Specify the control necessary to maintain the system in that state. Briefly discuss your result.

Solution: To maintain the system in that state, we need to have $x(k+1) = x(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ for $k \geq 3$. In other words,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.3 & 1 \\ -0.2 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0.50 \\ 0.25 \end{bmatrix} u(k),$$

for $k \geq 3$. Similarly, the above matrix equation has two individual equations, the first one gives $u(k) = -0.3/0.5$, whereas the second one gives $u(k) = 0.7/0.25$. Since there is no single value of $u(k)$ that can satisfy both of the equations, we conclude that there is no solution for the above matrix equation or the final state can't be maintained at $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. However, $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ can be reached in every two steps, since the system is reachable.

3. A discrete-time linear control system is described by

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -0.08 & 0.9 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 0.5 & 1 \end{bmatrix} \mathbf{x}(k),$$

where u, $\mathbf{x}$, and y are the input, the state, and the output variables, respectively, and a sampling period $T = 0.1\,\mathrm{s}$.

(a) Design a state-feedback controller, such that the following conditions are satisfied.

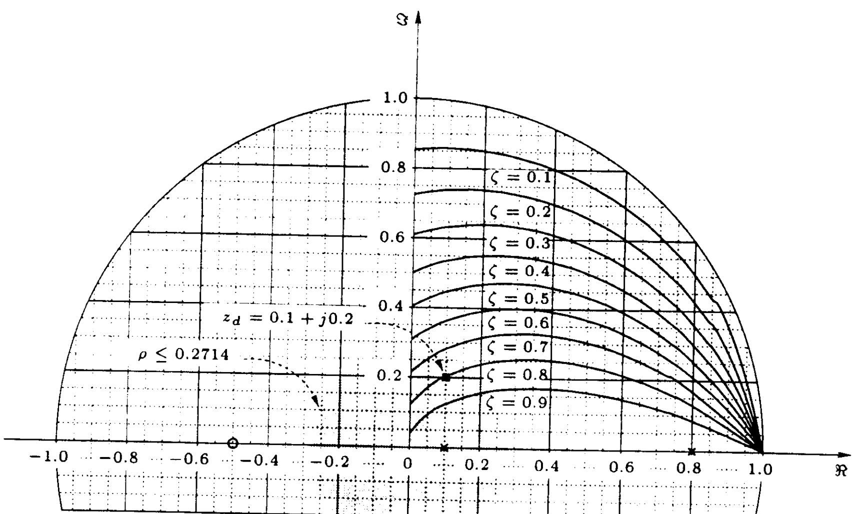
i. The maximum percent overshoot is between 1% and 3% for a unit-step input.

ii. The 2% settling time is less than 0.4 second.

Solution: We determine the restrictions on the location of the desired pole locations from the performance specifications.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent overshoot for a unit-step input	$0.01 < M_p < 0.03.$	From the α-M_p curves, $\zeta = 0.8$ provides the broadest range of α values, where $-80° < \alpha < 5°$.
Settling time for a unit-step input	$\rho \leq (0.02)^{1/(k_{2\%s}-1)}$.	For $t_{2\%s} = k_{2\%s}T \leq 0.4\,\text{s}$, and $k_{2\%s} \leq 0.4/0.1 = 4$, when $T = 0.1\,\text{s}$; $\rho \leq (0.02)^{1/(4-1)} = 0.2714.$

When we mark these restrictions on the z-plane, we determine that a possible set of desired-pole locations is at $z_d \approx 0.1 \pm j0.2$.



$$z = \Re(z) + j\Im(z) = e^{-\zeta\omega T}e^{j\sqrt{1-\zeta^2}\omega T}$$

Based on our choice of the desired-pole locations, the desired characteristic polynomial is given by

$$q_{c_d}(z) = \big(z - (0.1 + j0.2)\big)\big(z - (0.1 - j0.2)\big) = z^2 - 0.2z + 0.05.$$

We know that if a system is controllable, we can place the closed-loop poles at any desired location via state-feedback control. So assume

$$u(k) = K\mathbf{x}(k) = \begin{bmatrix} k_1 & k_2 \end{bmatrix}\mathbf{x}(k)$$

for some state-feedback matrix K. The characteristic polynomial of the system under state-feedback control can be determined from the denominator of the transfer function, such that

$$\begin{aligned} q_c(z) &= \det\big(zI - (A + BK)\big) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.08 & 0.9 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}\right)\right) \\ &= z^2 + (-k_2 - 0.9)z + (-k_1 + 0.08). \end{aligned}$$

Setting $q_c(z) = q_{c_d}(z)$, we get

$$-k_1 + 0.08 = 0.05,$$

or $k_1 = 0.03$; and

$$-k_2 - 0.9 = -0.2,$$

or $k_2 = -0.7$. Therefore,

$$u(k) = \begin{bmatrix} 0.03 & -0.7 \end{bmatrix}\mathbf{x}(k).$$

(b) Design the necessary additions to the controller assuming that only the output is available.

Solution: When only the output is available, state-feedback control can still be implemented if an observer is used. Moreover, we know that if a system is reachable, we can place the closed-loop poles of the observer at any desired location via error-feedback control. So assume

$$\mathbf{e}(k) = L(\hat{y}(k) - y(k)) = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}(\hat{y}(k) - y(k))$$

for some observer-error gain matrix L, where $\hat{y}$ is the observer output variable. Assuming that the observer poles are at 0.1 and 0.1, the desired observer-characteristic polynomial

$$q_{o_d}(z) = (z - 0.1)(z - 0.1) = z^2 - 0.2z + 0.01.$$

The observer-characteristic polynomial q_o under the error-feedback control can be determined from the denominator of the transfer function of the observer, such that

$$\begin{aligned} q_o(z) &= \det\big(zI - (A + LC)\big) \\ &= \det\left(z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ -0.08 & 0.9 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 0.5 & 1 \end{bmatrix}\right)\right) \\ &= z^2 + (-0.5l_1 - l_2 - 0.9)z + (0.53l_1 - 0.5l_2 + 0.08). \end{aligned}$$

Setting $q_o(z) = q_{o_d}(z)$, we get

$$-0.5l_1 - l_2 - 0.9 = -0.2,$$

and

$$0.53l_1 - 0.5l_2 + 0.08 = 0.01.$$

In matrix form, we get

$$\begin{bmatrix} -0.5 & -1 \\ 0.53 & -0.5 \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 0.7 \\ -0.07 \end{bmatrix},$$

and after solving for $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$, we obtain

$$\mathbf{e}(k) = \begin{bmatrix} 1 \\ -1.2 \end{bmatrix} \big(\hat{y}(k) - y(k)\big),$$

where $\mathbf{e}$ and $\hat{y}$ are the error-feedback control and the observer output variables, respectively.

4. Consider a system described by the difference equation

$$x(k+1) = -x(k) + u(k),$$

where x and u are the state and the input variables, respectively. Determine the optimal control action $u(k)$ for $k \geq 0$ that would minimize the cost function

$$J = \frac{1}{2}\sum_{k=0}^{2}\big(x^2(k) + 4u^2(k)\big),$$

when $x(0) = -1$ and $x(3) = 0$.

Solution: The Hamiltonian for this cost function and the system is

$$H_k\big(x(k), u(k), \lambda^*(k+1)\big) = \frac{1}{2}\big(x^2(k) + 4u^2(k)\big) + \lambda^T(k+1)\big(-x(k) + u(k)\big),$$

where λ is the Langrange multiplier. The optimality conditions in terms of the Hamiltonian are

$$\lambda(k) = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial x(k)} = x(k) - \lambda(k+1) \text{ for } 0 \leq k \leq 2,$$

$$0 = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial u(k)} = 4u(k) + \lambda(k+1) \text{ for } 0 \leq k \leq 2$$

$$x(k+1) = \frac{\partial H_k\big(x(k), u(k), \lambda(k+1)\big)}{\partial \lambda(k+1)} = -x(k) + u(k) \text{ for } 0 \leq k \leq 2.$$

From the above optimality equations, we get

$$\lambda(k+1) = x(k) - \lambda(k),$$

and

$$\begin{aligned} x(k+1) &= -x(k) + u(k) = -x(k) + \big(-(1/4)\lambda(k+1)\big) \\ &= -x(k) - (1/4)\big(x(k) - \lambda(k)\big) = -(5/4)x(k) + (1/4)\lambda(k). \end{aligned}$$

Or, in matrix form

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}.$$

Since the boundary conditions $x(0) = -1$ and $x(3) = 0$ are given, we need to solve the above matrix equation to determine $\lambda(0)$. Since,

$$\begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix}^2 = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 29/16 & -(9/16) \\ -(9/4) & 5/4 \end{bmatrix},$$

$$\begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix}^3 = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix}^2 \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -(181/64) & 65/64 \\ 65/16 & -(29/16) \end{bmatrix},$$

and

$$\begin{bmatrix} x(3) \\ \lambda(3) \end{bmatrix} = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix}^3 \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix};$$

we get

$$x(3) = -(181/64)x(0) + (65/64)\lambda(0),$$
$$0 = -(181/64)(-1) + (65/64)\lambda(0),$$

or $\lambda(0) = -(181/65) = -2.7842$.

Since $u(k) = -(1/4)\lambda(k+1)$ for $k = 0, 1, 2$ from the optimality condition, we need to determine $\lambda(k)$ for $k = 1, 2, 3$.

$$\begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ \lambda(0) \end{bmatrix} = \begin{bmatrix} 36/65 \\ 116/65 \end{bmatrix}.$$

$$\begin{bmatrix} x(2) \\ \lambda(2) \end{bmatrix} = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(1) \\ \lambda(1) \end{bmatrix} = \begin{bmatrix} -16/65 \\ -16/13 \end{bmatrix}.$$

$$\begin{bmatrix} x(3) \\ \lambda(3) \end{bmatrix} = \begin{bmatrix} -(5/4) & 1/4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(2) \\ \lambda(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 64/65 \end{bmatrix}.$$

From $u(k) = -(1/4)\lambda(k+1)$ for $k = 0, 1, 2$, we get

$$u(0) = -(29/65), \quad u(1) = 4/13, \quad \text{and } u(2) = -(16/65).$$