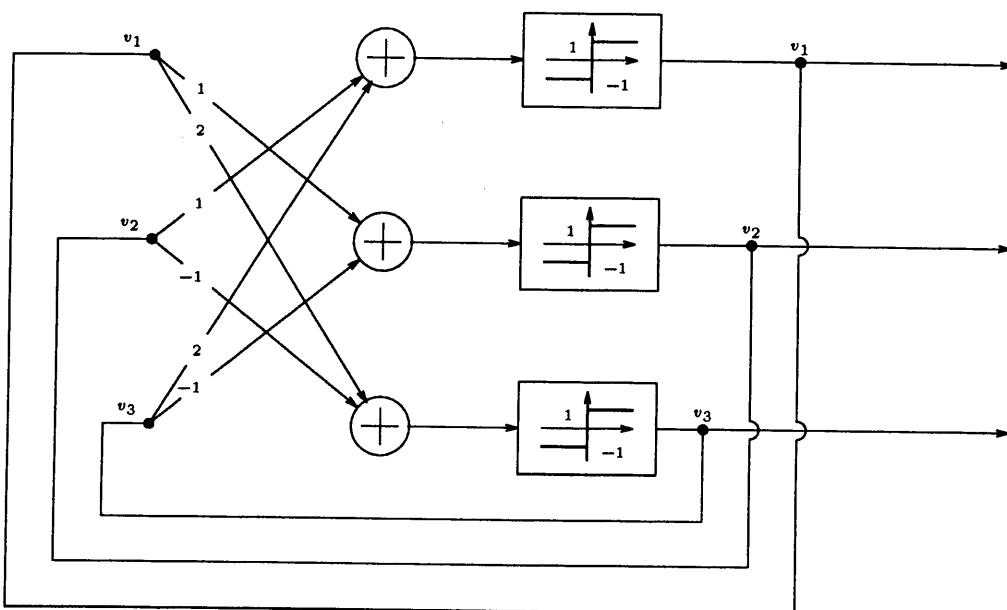


1. Consider the following discrete-time recurrent network with bipolar neurons.



- (a) Determine the weight matrix W from the figure by inspection. (5pts)
- (b) Determine the energy function based on the weight matrix in terms of the states, and calculate the energy value at each state. Note: The energy values of conjugate pairs are identical. (5pts)
- (c) Based on the energy values obtained above, identify equilibrium states. (5pts)
- (d) Analyze the synchronous updates and the corresponding energy value updates starting from the initial state $v = [1, -1, -1]^T$. (5pts)
- (e) Analyze the asynchronous updates and the corresponding energy value updates starting from the same initial state $v = [1, -1, -1]^T$. Assume the updates take place in natural order starting with the first neuron. (5pts)

2. The dynamical equations of a nonlinear system are given by

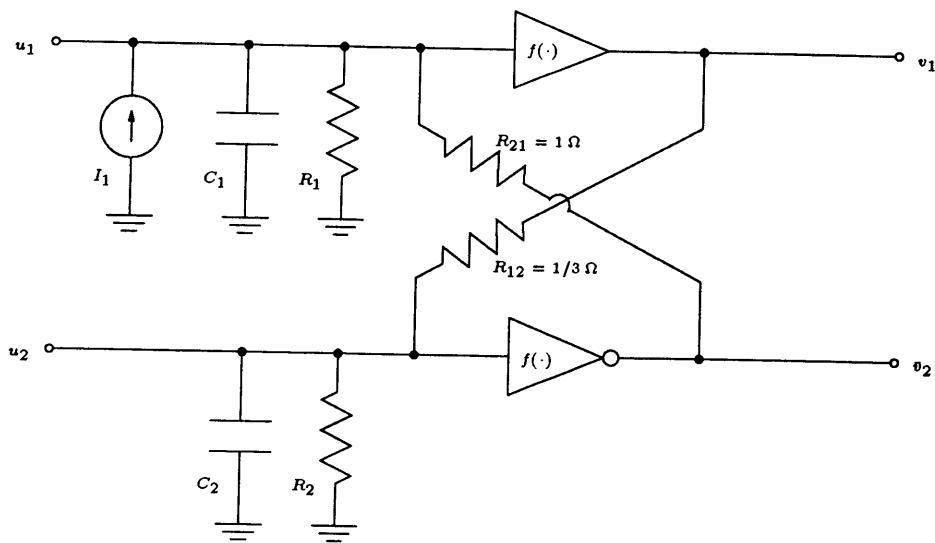
$$\begin{aligned}\dot{x}_1 &= (x_2 - 1)x_1^3, \\ \dot{x}_2 &= -\frac{x_1^4}{(1 + x_1^2)^2} - \frac{x_2}{1 + x_2^2}.\end{aligned}$$

A candidate for a Lyapunov function for the system is

$$L(x_1, x_2) = 1 - \frac{1}{1 + x_1^2} + x_2^2.$$

- (a) Check if L is a proper Lyapunov function. (10pts)
- (b) Give possible conclusions about the system stability from the previous part if possible. (10pts)

3. Consider the following continuous-time single layer feedback network.



- (a) Obtain the state equations. (10pts)
 - (b) Determine the weight matrix W . (5pts)
 - (c) Determine the truncated energy function for high-gain neurons. (5pts)
 - (d) Find the symmetric weight matrix W' that would give the same energy, and the corresponding values of resistors R'_{21} and R'_{12} . (5pts)
4. A continuous-time control plant has an unknown parameter α in its dynamics given by

$$\ddot{y} + 5\dot{y} + 2y = \alpha\dot{u} + 3u.$$

An identifier is to be designed for this plant to identify the unknown parameter α . The identifier minimizes an incremental error cost function which is given by

$$J = \frac{1}{2} \int_{(k-1)T}^{kT} (y - \hat{y})^2 dt,$$

where y is the output of the plant, and \hat{y} is the output of the identifier.

- (a) Obtain a sensitivity model of the plant with respect to α , and draw the block diagram of the sensitivity model in controller canonical form. (10pts)
- (b) Obtain an update equation for α which iteratively minimizes the error cost function J . (10pts)
- (c) Multi-layer feedforward neural networks with error back-propagation training are decided to be used for the identifier and the sensitivity model. Draw the block diagram showing the connections among the plant, the identifier, and the sensitivity model. Clearly mark all the inputs and outputs of the blocks, and show the signals needed for the training of the networks. On the block diagram, represent the plant, the identifier, and the sensitivity model by single blocks without giving their internal details. (10pts)

#1 a)

$$W = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$b_1, F = -\frac{1}{2}v^T W v$$

$$= -\frac{1}{2} [v_1 \ v_2 \ v_3] \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$= -\frac{1}{2} [v_1 \ v_2 \ v_3] \begin{bmatrix} v_2 + 2v_3 \\ v_1 - v_3 \\ 2v_1 - v_2 \end{bmatrix}$$

$$= -\frac{1}{2} (v_1 v_2 + 2v_1 v_3 + v_1 v_2 - v_2 v_3 + 2v_1 v_3 - v_2 v_3)$$

$$F = -v_1 v_2 - 2v_1 v_3 + v_2 v_3$$

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow F = -1 - 2 + 1 = -2 \Leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \check{v}$$

$$v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow F = -1 + 2 - 1 = 0 \Leftrightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \check{v}$$

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow F = 1 - 2 - 1 = -2 \Leftrightarrow \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \check{v}$$

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow F = 1 + 2 + 1 = 4 \Leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \check{v}$$

c// Equilibrium states are when energy is minimum.

Check $v = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ & $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\underline{\underline{v}} = \underline{\underline{Wv}} = \underline{\underline{V(t)}}$$

$$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \leftarrow \text{Equilibrium State}$$

$$\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow \text{Equilibrium State}$$

d// Synchronous update from $v = [1 \ -1 \ -1]^T$

$$\underline{\underline{v}} = \underline{\underline{Wv}} = \underline{\underline{V(t)}}$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \leftarrow \begin{array}{l} \text{Starting} \\ \text{State} \end{array}$$

e,, Asynchronous update

$$\underline{\underline{v}} = \underline{\underline{Wv}} = \underline{\underline{V(t)}}$$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \leftarrow \begin{array}{l} \text{Equilibrium} \\ \text{State} \end{array}$$

+
Convergent

#2 Dynamics

$$\dot{x}_1 = (x_2 - 1)x_1^3$$

$$\dot{x}_2 = -\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2}$$

Lyapunov
function

$$L(x_1, x_2) = 1 - \frac{1}{1+x_1^2} + x_2^2$$

Q1: L is a Lyapunov function if

i) $L(x_1, x_2) > 0 \quad \|x\| > 0$

$L(x_1, x_2) = 0 \quad \|x\| = 0$

ii) $\frac{dL(x_1, x_2)}{dt} < 0$

Check i: $L(x_1, x_2) = 1 - \frac{1}{1+x_1^2} + x_2^2$

$$= \frac{1+x_1^2-1}{1+x_1^2} + x_2^2$$

$$= \frac{x_1^2}{1+x_1^2} + x_2^2 > 0 \quad \|x\| > 0$$

since all terms are positive

$$= 0 \quad \|x\| = 0$$

Check ii: $\frac{dL(x_1, x_2)}{dt} = \frac{2x_1 \dot{x}_1}{(1+x_1^2)^2} + 2x_2 \dot{x}_2$

$$= \frac{2x_1(x_2 - 1)x_1^3}{(1+x_1^2)^2} + 2x_2 \left(-\frac{x_1^4}{(1+x_1^2)^2} - \frac{x_2}{1+x_2^2} \right)$$

$$= 2 \frac{x_1^4 x_2}{(1+x_1^2)^2} - 2 \frac{x_1^4}{(1+x_1^2)^2} - 2 \frac{x_1^4 x_2}{(1+x_2^2)}$$

$$\begin{aligned}
 & -2 \frac{x_2^2}{1+x_2^2} \\
 & = -2 \frac{x_1^4}{(1+x_1^2)^2} - 2 \frac{x_2^2}{1+x_2^2} \leq 0 \quad \text{for } \|x\| > 0 \\
 & \quad \text{since all terms are neg.}
 \end{aligned}$$

b) From the previous part, $L(x_1, x_2)$ terms set to be a proper Lyapunov fnct. Therefore, system is locally stable around the equilibrium state.

Equilibrium state when $\dot{x}_1 = 0 \Rightarrow x_2 = 1$ or $x_1 = 0$
 $\dot{x}_2 = 0$

↓ ↓

$$\dot{x}_2 \neq 0 \quad \dot{x}_2 = 0 \Rightarrow x_2 = 0$$

since

$$\dot{x}_2 = -\frac{x_1^4}{(1+x_1^2)^2} - \frac{1}{2}$$

↑
Equilibrium

always non-pos.

In other words, system is locally stable around $x = [0 \ 1]^T$.

#3 a) Node eqns.

$$-\mathcal{D}_1 + C_1 \frac{du_1}{dt} + \frac{1}{R_1} u_1 + \frac{1}{R_{21}} (u_1 - v_2) = 0$$

$$C_2 \frac{du_2}{dt} + \frac{1}{R_2} u_2 + \frac{1}{R_{12}} (u_2 - v_1) = 0$$

or $C_1 \frac{du_1}{dt} = \frac{1}{R_{21}} v_2 - \left(\frac{1}{R_{21}} + \frac{1}{R_1} \right) u_1 + \mathcal{D}_1$

$$C_2 \frac{du_2}{dt} = \frac{1}{R_{12}} v_1 - \left(\frac{1}{R_{12}} + \frac{1}{R_2} \right) u_2$$

or $C_1 \frac{du_1}{dt} = -v_2 - \left(1 + \frac{1}{R_1} \right) u_1 + \mathcal{D}_1$

$$C_2 \frac{du_2}{dt} = 3v_1 - \left(3 + \frac{1}{R_2} \right) u_2$$

b) From the state eqns.

$$\omega = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$$

$$e_A E = -\frac{1}{2} v^T \omega v - i^T v$$

$$= -\frac{1}{2} [v_1 \ v_2] \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - [\mathcal{D}_1 \ \mathcal{D}_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= -\frac{1}{2} [v_1 \ v_2] \begin{bmatrix} -v_2 \\ 3v_1 \end{bmatrix} - \mathcal{D}_1 v_1$$

$$= -\frac{1}{2} (-v_1 v_2 + 3v_1 v_2) - \mathcal{D}_1 v_1 = -v_1 v_2 - \mathcal{D}_1 v_1$$

$$\text{def } \boldsymbol{w}' = \frac{1}{2} (\boldsymbol{w} + \boldsymbol{w}^T) = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

In the original weight matrix

$$\boldsymbol{w} = \begin{bmatrix} 0 & -\frac{1}{2}R_{21} \\ \frac{1}{2}R_{12} & 0 \end{bmatrix}$$

$$\text{So } \boldsymbol{w}' = \begin{bmatrix} 0 & -\frac{1}{2}R'_{21} \\ \frac{1}{2}R'_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow R'_1 = 1 \text{ and } R'_{21} = -1$$

or use v_2 instead of \bar{v}_2

$$\Rightarrow R'_1 = 1 \text{ and } R'_{21} = 1$$

$$\#4 \text{ Dynamics: } \ddot{y} + 5\dot{y} + 2y = \alpha \ddot{u} + 3u$$

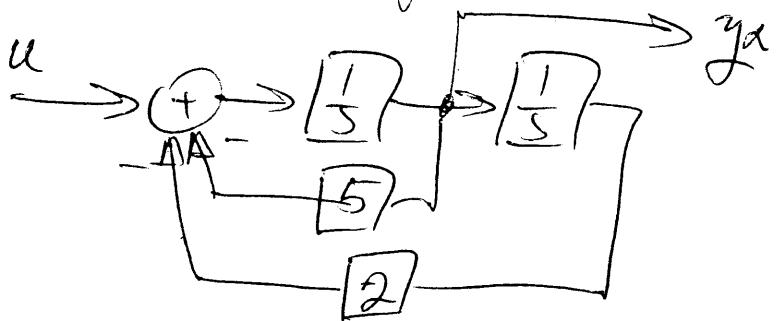
$$\text{Error: } J = \frac{1}{2} \int_{(k-1)\tau}^{k\tau} (y - \hat{y})^2 dt$$

or Sensitivity model is

$$\frac{\partial}{\partial \alpha} (\ddot{y} + 5\dot{y} + 2y = \alpha \ddot{u} + 3u)$$

$$\ddot{y}\alpha + 5\dot{y}\alpha + 2y\alpha = \ddot{\alpha}$$

Controller canonical form



b) Update eqn. for α from $\frac{d\hat{\alpha}}{d\alpha}$

$$\begin{aligned}\frac{d\hat{\alpha}}{d\alpha} &= \frac{1}{2} \int_{(k-1)\tau}^{k\tau} \frac{d(y - \hat{y})^2}{d\alpha} dt \\ &= \frac{1}{2} \int_{(k-1)\tau}^{k\tau} 2(y - \hat{y}) \left(-\frac{dy}{d\alpha}\right) dt \\ &= - \int_{(k-1)\tau}^{k\tau} (y - \hat{y}) \hat{y}_a dt\end{aligned}$$

$$\text{So } \alpha(t) = \alpha - \int \frac{d\hat{\alpha}}{d\alpha}$$

$$= \alpha + \int_{(k-1)\tau}^{k\tau} (y - \hat{y}) \hat{y}_a dt$$

CII

