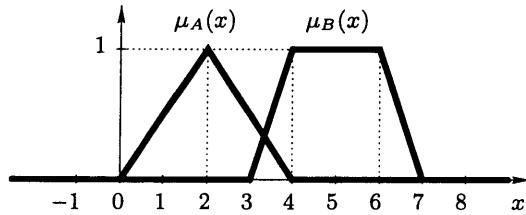


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1. Consider two fuzzy sets  $A$  and  $B$  with the following membership functions,  $\mu_A$  and  $\mu_B$ , respectively.



Determine  $(A \cap B) \cup \bar{A}$  and sketch its membership function, where

$$\begin{aligned}\mu_{A \cap B} &= \max(0, \mu_A + \mu_B - 1), \\ \mu_{A \cup B} &= \min(1, \mu_A + \mu_B),\end{aligned}$$

and

$$\mu_{\bar{A}} = 1 - \mu_A.$$

(20pts)

2. Consider the fuzzy set  $A$  with the membership function

$$\mu_A(x) = \begin{cases} x+1, & \text{if } -1 < x < 0; \\ x, & \text{if } 0 \leq x < 1; \\ -x+2, & \text{if } 1 \leq x < 2; \\ 0, & \text{otherwise.} \end{cases}$$

Determine the membership function of  $A^2$ .

(20pts)

3. Consider the generalized modus ponens for fuzzy sets, such that

$$\frac{\begin{array}{c} A' \\ A \rightarrow B \\ \hline B' \end{array}}{ }$$

for fuzzy sets  $A$ ,  $B$ ,  $A'$ , and  $B'$ , where  $A'$  and  $B'$  are strongly related to  $A$  and  $B$ , respectively. Determine the consequence  $B'$ ; when  $A'$  is very- $A$  under the Dienes-Rescher implication, such that

$$\mu_{A \rightarrow B}(x, y) = \sup(1 - \mu_A(x), \mu_B(y))$$

and the  $t$ -function is the infimum, where  $\mu_A$ ,  $\mu_B$ , and  $\mu_{A \rightarrow B}$  are the membership functions of the sets  $A$ ,  $B$ , and  $A \rightarrow B$ , respectively.

(30pts)

4. Consider a single-rule fuzzy logic system, such that

$$\mathcal{R} : A \longrightarrow B,$$

where  $A$  and  $B$  are fuzzy sets with the membership functions

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \leq 0; \\ 1 - x, & \text{if } 0 < x \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$

and

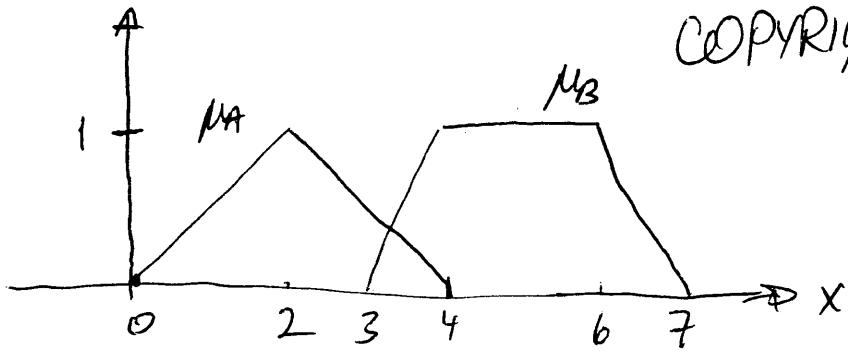
$$\mu_B(y) = \begin{cases} 0, & \text{if } y \leq 1; \\ y - 1, & \text{if } 1 < y \leq 2; \\ 1, & \text{if } 2 < y, \end{cases}$$

respectively. Assume the input is a fuzzy set  $A'$ , that is strongly related to  $A$ , with the membership function

$$\mu_{A'}(x) = \begin{cases} 1 - |x|, & \text{if } -1 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Determine the corresponding output fuzzy set  $B'$  by assuming the Lukasiewicz inference engine. In other words, assume the individual-based inference, the intersection logic resolution, the Lukasiewicz implication, and the infimum as the  $t$ -function. (30pts)

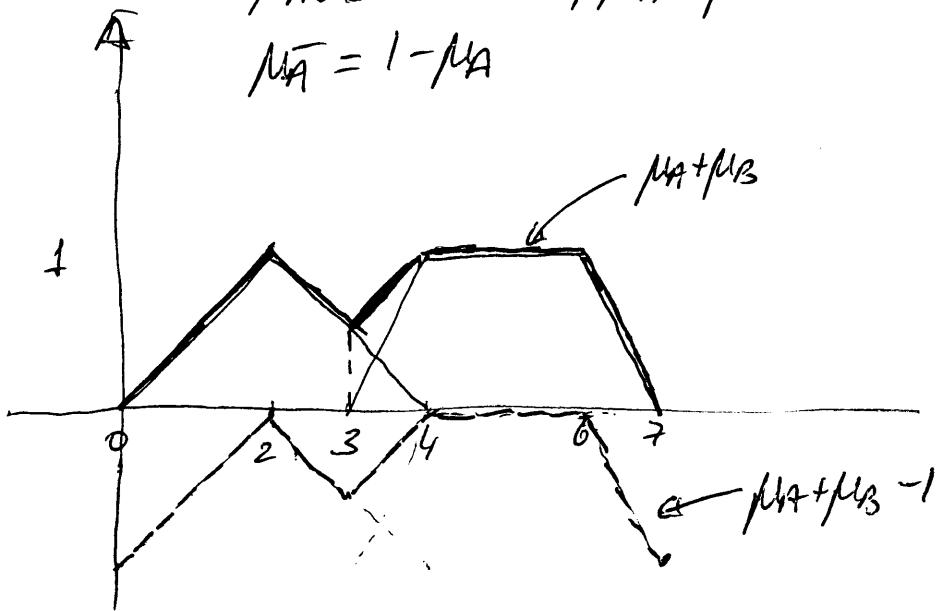
#1

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$$\mu_{A \cap B} = \max(0, \mu_A + \mu_B - 1)$$

$$\mu_{A \cup B} = \min(1, \mu_A + \mu_B)$$

$$\bar{\mu}_A = 1 - \mu_A$$



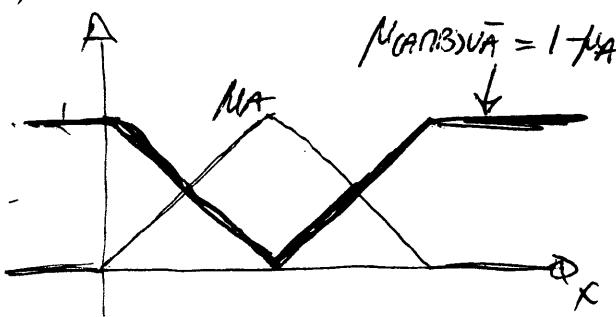
$$\mu_{A \cap B}(x) = \max(0, \mu_A + \mu_B - 1) = 0$$

$$\mu_{(A \cap B) \cup \bar{A}}(x) = \min(1, \mu_{A \cap B}(x) + \bar{\mu}_A(x))$$

$$= \min(1, \bar{\mu}_A(x))$$

$$= \min(1, 1 - \mu_A(x))$$

$$= 1 - \mu_A(x)$$



## EXAM #1 SOLUTION

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#2

$$\mu_A(x) = \begin{cases} x+1, & \text{if } -1 \leq x < 0; \\ x, & \text{if } 0 \leq x < 1; \\ -x+2, & \text{if } 1 \leq x \leq 2; \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{A^2}(y) = \sup_{y=x^2} (\mu_A(x))$$

$$\textcircled{1} \quad -1 \leq x < 0, \quad y = x^2 \Rightarrow x = -y^{1/2}$$

$$-1 \leq -y^{1/2} < 0$$

$$0 \leq y^{1/2} < 1$$

$0 \leq y < 1$  since  $y^2$  is an increasing function of  $y$  for  $y > 0$ .

$$\mu_A(x) / \underset{x=-y^{1/2}}{=} -y^{1/2} + 1$$

$$\textcircled{2} \quad 0 \leq x < 1, \quad y = x^2 \Rightarrow x = y^{1/2}$$

$$0 \leq y^{1/2} < 1$$

$0 \leq y < 1$

$$\mu_A(x) / \underset{x=y^{1/2}}{=} y^{1/2}$$

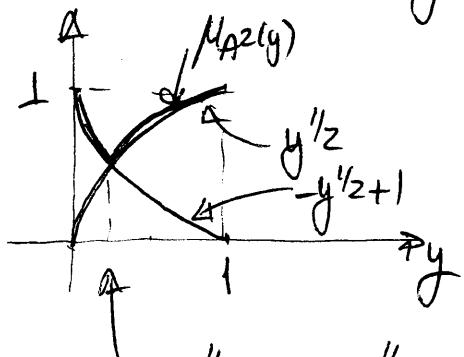
$$\textcircled{3} \quad 1 \leq x \leq 2, \quad y = x^2 \Rightarrow x = y^{1/2}$$

$$\begin{array}{|c|} \hline 1 \leq y^{1/2} \leq 2 \\ \hline 1 \leq y \leq 4 \\ \hline \end{array}$$

$$\mu_{A^2}(x) / \begin{cases} x = y^{1/2} \end{cases} = -y^{1/2} + 2$$

So from \textcircled{1} \& \textcircled{2}  $0 \leq y \leq 1$

$$\mu_{A^2}(y) = \sup_y (-y^{1/2} + 1, y^{1/2})$$



$$-y^{1/2} + 1 = y^{1/2} \Rightarrow y^{1/2} = \frac{1}{2}, \quad y = \frac{1}{4}$$

$$\mu_{A^2}(y) = \begin{cases} -y^{1/2} + 1, & 0 \leq y \leq \frac{1}{4} \\ y^{1/2}, & \frac{1}{4} < y \leq 1 \end{cases}$$

From \textcircled{3}  $1 \leq y \leq 4$

$$\mu_{A^2}(y) = \sup_y (-y^{1/2} + 2) = -y^{1/2} + 2$$

# Exam #1 Solutions

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Then  $\mu_{A^2}(y) = \begin{cases} -y^{1/2} + 1, & 0 \leq y \leq 1/4 \\ y^{1/2}, & 1/4 \leq y \leq 1 \\ -y^{1/2} + 2, & 1 \leq y \leq 4 \\ 0, & \text{otherwise} \end{cases}$

#3

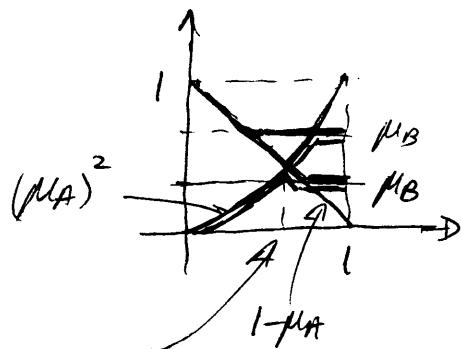
$$\frac{\theta'}{A \rightarrow B}$$

let  $A' = \text{very-}A \Rightarrow \mu_{A'}(x) = (\mu_A(x))^2$

Dienes-Roscher  
Duplication

$$\Rightarrow \mu_{A \rightarrow B}(x, y) = \sup(1 - \mu_A(x), \mu_B(y))$$

$$\begin{aligned} \mu_B(y) &= \sup_x t(\mu_{A'}(x), \mu_{A \rightarrow B}(x, y)) \\ &= \sup_x (\inf((\mu_A(x))^2, \sup(1 - \mu_A(x), \mu_B(y)))) \end{aligned}$$



$$(\mu_A)^2 = 1 - \mu_A \Rightarrow (\mu_A)^2 + \mu_A - 1 = 0, 0 \leq \mu_A \leq 1$$

$$\mu_A = \frac{-1 + \sqrt{1+4}}{2} = \frac{1}{2}(\sqrt{5}-1)$$

$$\text{Value at } = 1 - \mu_A | = \frac{1}{2} (3 - \sqrt{5})$$

$$\mu_A = \frac{1}{2} (\sqrt{5} - 1) \quad \mu_A = \frac{1}{2} (\sqrt{5} - 1)$$

If  $\mu_B(y) > \frac{1}{2} (3 - \sqrt{5})$ , then  $\mu_{B'}(y) = \mu_B(y)$

If  $\mu_B(y) \leq \frac{1}{2} (3 - \sqrt{5})$ , then  $\mu_{B'}(y) = \frac{1}{2} (3 - \sqrt{5})$

$$\Rightarrow \mu_{B'}(y) = \max\left(\frac{1}{2} (3 - \sqrt{5}), \mu_B(y)\right)$$

#4

$$R : A \rightarrow B$$

$$\mu_A(x) = \begin{cases} 1, & x \leq 0 \\ 1-x, & 0 < x \leq 1 \\ 0, & \text{otherwise} \end{cases} (1 \leq x)$$

$$\mu_B(y) = \begin{cases} 0, & y \leq 1 \\ y^{-1}, & 1 < y \leq 2 \\ 1, & 2 < y \end{cases}$$

$$\mu_{A'}(x) = \begin{cases} 1 - |x|, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Lukasiewicz inference engine

$$\mu_{B'}(y) = \min \left( \sup_x \left( \min \left( \mu_{A'_1}(x_1), \dots, \mu_{A'_n}(x_n) \right), 1 - \min \left( \mu_{A'_1}(x_1), \dots, \mu_{A'_n}(x_n) \right) + \mu_B(y) \right) \right)$$

## Exam #1 Review

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In our case

$$\mu_{B'}(y) = \sup_x (\min(\mu_A(x), 1 - \mu_A(x) + \mu_B(y)))$$

$$1 - \mu_A(x) + \mu_B(y) = \begin{cases} 1 - 1 + 0, & x \leq 0, y \leq 1 \\ 1 - 1 + y - 1, & x \leq 0, 1 < y \leq 2 \\ 1 - 1 + 1, & x \leq 0, 2 < y \\ 1 - (1-x) + 0, & 0 < x \leq 1, y \leq 1 \\ 1 - (1-x) + (y-1), & 0 < x \leq 1, 1 < y \leq 2 \\ 1 - (1-x) + 1, & 0 < x \leq 1, 2 < y \\ 1 - 0 + 0, & 1 < x, y \leq 1 \\ 1 - 0 + (y-1), & 1 < x, 1 < y \leq 2 \\ 1 - 0 + 1, & 1 < x, 2 < y \end{cases}$$

$$= \begin{cases} 0, & x \leq 0, y \leq 1 \\ y - 1, & x \leq 0, 1 < y \leq 2 \\ 1, & x \leq 0, 2 < y \\ x, & 0 < x \leq 1, y \leq 1 \\ x + y - 1, & 0 < x \leq 1, 1 < y \leq 2 \\ x + 1, & 0 < x \leq 1, 2 < y \\ 1, & 1 < x, y \leq 1 \\ y, & 1 < x, 1 < y \leq 2 \\ 2, & 1 < x, 2 < y \end{cases}$$

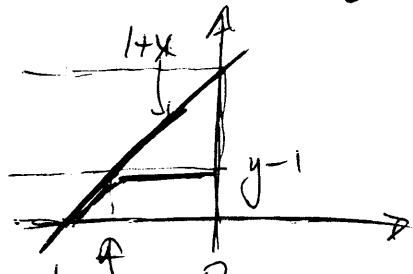
# EXAM #1 SOLUTIONS

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min

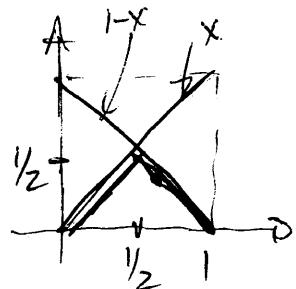
$1+x+y$	0	$y-1$	1	$x$	$x+y-1$	$x+1$	1	$y$	2
$x \leq 0$	$x \leq 0$	$x \leq 0$	$0 \leq x \leq 1$	$0 \leq x \leq 1$	$0 \leq x \leq 1$	$1 \leq x$	$1 \leq x$	$1 \leq x$	$1 \leq x$
$y \leq 1$	$1 \leq y \leq 2$	$2 \leq y$	$y \geq 1$	$1 \leq y \leq 2$	$2 \leq y$	$y \leq 1$	$1 \leq y \leq 2$	$2 \leq y$	
0	0	0	0	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /
-1	0	0	□	1+x	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /
$1+x$	0	0	□	1+x	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /
$1-x$	0	0	0	/ / / /	□	□	□	/ / / /	/ / / /
1	0	0	0	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /	/ / / /
0	0	0	0	/ / / /	0	0	0	0	0

$\min(1+x, y-1) = \begin{cases} 1+x, & -1 \leq x \leq y-2 \\ y-1, & y-2 \leq x \leq 0 \end{cases}$



$$1+x=y-1 \Rightarrow x=y-2$$

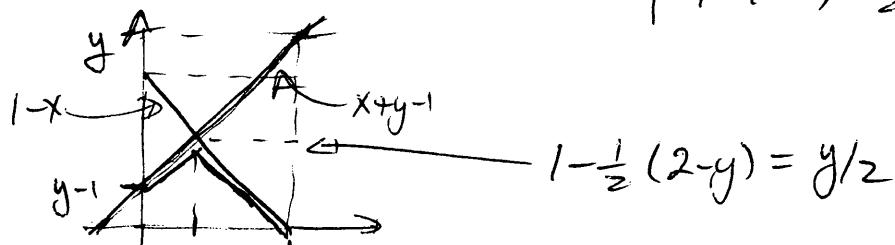
$\min(1-x, x) = \begin{cases} x, & 0 \leq x \leq 1/2 \\ 1-x, & 1/2 \leq x \leq 1 \end{cases}$



EXAM #1 Solutions

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3  $\min(1-x, x+y-1) = \begin{cases} x+y-1, & 0 < x < \frac{1}{2}(2-y) \\ 1-x, & \frac{1}{2}(2-y) \leq x \leq 1 \end{cases}$



$$1-x = x+y-1, \quad 2x = 2-y, \quad x = \frac{1}{2}(2-y)$$

4  $\min(1-x, x+1) = 1-x \text{ for } 0 < x < 1$

For  $y \leq 1$ ;  $\min(\dots) = 0, x \leq 0$

$$\min(\dots) = \begin{cases} x, & 0 < x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

2

$$\min(\dots) = 0, 1 < x$$

$$\sup_x (\min(\dots)) = \frac{1}{2} \quad (\text{when } x = \frac{1}{2})$$

For  $1 \leq y \leq 2$ ;  $\min(\dots) = 0, x \leq -1$

$$\min(\dots) = \begin{cases} 1+x, & -1 < x < y-2 \\ y-1, & y-2 \leq x \leq 0 \end{cases}$$

1

$$\min(\dots) = \begin{cases} x+y-1, & 0 < x < \frac{1}{2}(2-y) \\ 1-x, & \frac{1}{2}(2-y) \leq x < 1 \end{cases}$$

3

$$\min(\dots) = 0, 1 < x$$

$$\sup_x (\min(\dots)) = \max(y-1, y/2) \quad \begin{array}{l} \text{(when } x = y-2) \\ \text{(when } x = \frac{1}{2}(2-y)) \end{array}$$

# EXAM #1 SOLUTIONS

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for  $2 \leq y$ ;  $\min(\dots) = 0$ ;  $x \leq -1$

$\min(\dots) = 1+x$ ;  $-1 < x \leq 0$

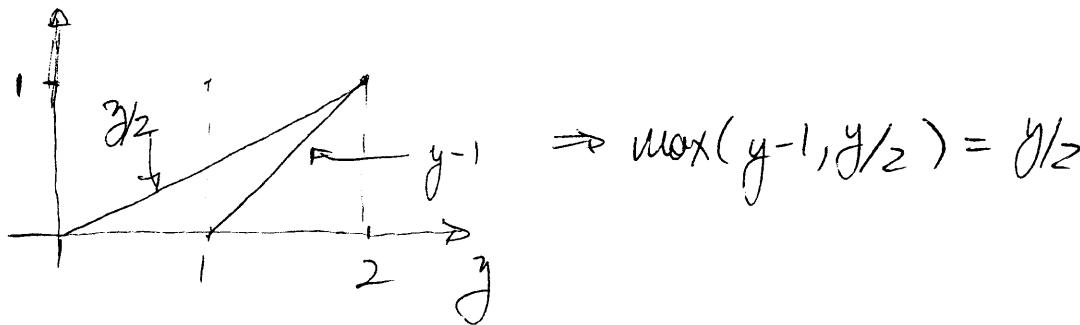
$\min(\dots) = 1-x$ ;  $0 < x \leq 1$

[4]

$\min(\dots) = 0$ ;  $1 < x$

$$\sup_x \min(\dots) = 1 \quad (\text{when } x=0)$$

$$\text{so } \mu_{B^1}(y) = \begin{cases} \frac{1}{2}, & y \leq 1 \\ \max(y-1, y/2), & 1 < y \leq 2 \\ 1, & 2 < y \end{cases}$$



$$\mu_{B^1}(y) = \begin{cases} \frac{1}{2}, & y \leq 1 \\ y/2, & 1 < y \leq 2 \\ 1, & 2 < y \end{cases}$$