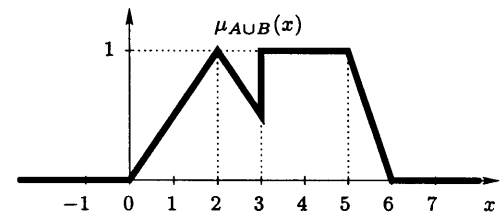1. Consider two fuzzy sets A and B with the following membership functions, μ_A and μ_B, respectively.
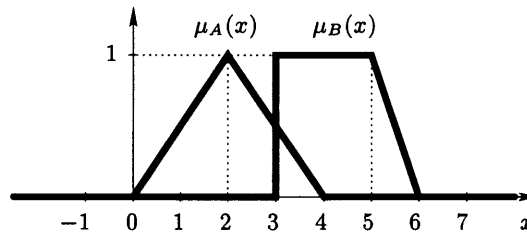
(a) Determine $A \cup B$, and sketch its membership function, where $\mu_{A \cup B} = \sup(\mu_A, \mu_B)$.

**Solution:** From the definition of the union, we get the following membership function for $A \cup B$.

$$\mu_{A \cup B}(x) = \begin{cases} (1/2)x, & \text{if } 0 < x \le 2; \\ -(1/2)(x-4), & \text{if } 2 < x \le 3; \\ 1, & \text{if } 3 < x \le 5; \\ -(x-6), & \text{if } 5 < x \le 6; \\ 0, & \text{otherwise.} \end{cases}$$


(b) Determine $A \cup B$, and sketch its membership function, where $\mu_{A \cup B} = \mu_A + \mu_B - \mu_A \mu_B$.

Solution: In this case, we could calculate the membership function from the analytical representations of the individual membership functions, or we could separate the expression for the union into manageable components, such as $\mu_A + \mu_B$ and $\mu_A \mu_B$.

The sum of the individual membership functions is easily obtained from the given sketchs.
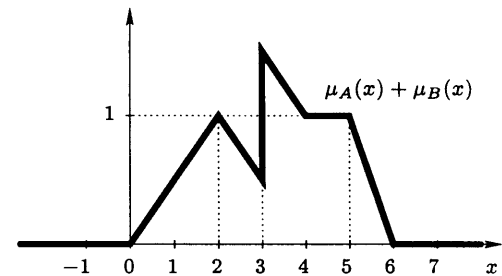
Similarly, the product of the individual membership functions is easily obtained from the given sketchs; since in most of the regions, one of the functions is zero.
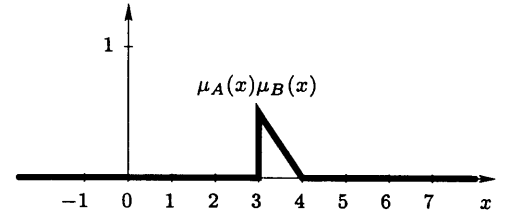
When we subtract the above product from the above sum, we get the sketch of the membership function of the union.

$$\mu_{A\cup B}(x) = \begin{cases} (1/2)x, & \text{if } 0 < x \leq 2; \\ -(1/2)(x-4), & \text{if } 2 < x \leq 3; \\ 1, & \text{if } 3 < x \leq 5; \\ -(x-6), & \text{if } 5 < x \leq 6; \\ 0, & \text{otherwise.} \end{cases}$$
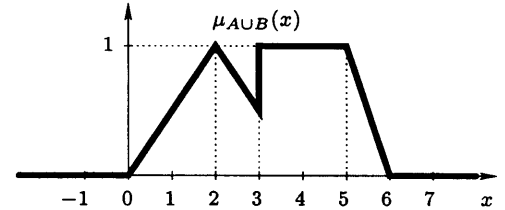
2. Consider a two-dimensional fuzzy set $A_1 \times A_2$ with the membership function

$$\mu_{A_1 \times A_2}(x_1, x_2) = e^{-(x_1+x_2)^2},$$

where $x_1 \in [-1, 1]$, and $x_2 \in [0, 1]$.

(a) Determine the projection of the fuzzy set $A_1 \times A_2$ on its first dimension, where the membership function of the projection is denoted by $\mu_{P_{A_1}}$.

Solution: From the definition of the projection, we have

$$\mu_{P_{A_1}}(x_1) = \sup_{x_2 \in [0,1]} \left(\mu_{A_1 \times A_2}(x_1, x_2)\right) = \sup_{0 \leq x_2 \leq 1} \left(e^{-(x_1+x_2)^2}\right).$$

Since the function $e^{-(x_1+x_2)^2}$ is continuous and bounded, its supremum with respect to x_2 in a compact domain is either at a point where the partial derivative is zero or on one of the boundaries. The partial derivative gives

$$\frac{\partial \mu_{A_1 \times A_2}(x_1, x_2)}{\partial x_2} = \frac{\partial (e^{-(x_1+x_2)^2})}{\partial x_2} = -2(x_1 + x_2)e^{-(x_1+x_2)^2};$$

and when $\partial \mu_{A_1 \times A_2}(x_1, x_2)/\partial x_2 = 0$, we get $x_2 = -x_1$. However, since $-1 \leq x_1 \leq 1$, but $0 \leq x_2 \leq 1$; when $0 \leq x_1 \leq 1$, x_2 cannot be equal to $-x_1$. In this case, the supremum value occurs at the boundary where $x_2 = 0$. Since $(e^{-(x_1+x_2)^2})_{x_2=-x_1} = 1$, and $(e^{-(x_1+x_2)^2})_{x_2=0} = e^{-x_1^2}$; we get

$$\mu_{P_{A_1}}(x_1) = \begin{cases} 1, & \text{if } -1 \leq x_1 \leq 0; \\ e^{-x_1^2}, & \text{if } 0 < x_1 \leq 1. \end{cases}$$

(b) Determine the cylindrical extension of the projection in the previous part back to the two-dimensional space, where the membership function of the cylindrical extension is denoted by $\mu_{P_{A_1} \times E_{A_2}}$.

Solution: From the definition of the extension, we have

$$\mu_{P_{A_1} \times E_{A_2}}(x_1, x_2) = \mu_{P_{A_1}}(x_1),$$

or

$$\mu_{P_{A_1} \times E_{A_2}}(x_1, x_2) = \begin{cases} 1, & \text{if } -1 \le x_1 \le 0; \\ e^{-x_1^2}, & \text{if } 0 < x_1 \le 1. \end{cases}$$

3. Consider the two fuzzy relations P and Q with the membership functions

$$\mu_P(x, y) = \min\big(1, \mu_A(x) + \mu_B(y)\big),$$

and

$$\mu_Q(y, z) = \max\big(0, \mu_C(z) - \mu_B(y)\big),$$

where μ_A, μ_B, and μ_C are membership functions of fuzzy sets A, B, and C, respectively. Determine the membership function $\mu_{P \circ Q}(x, z)$ of the composition $P \circ Q$, when the s-function is the supremum, and the t-function is the infimum. Assume that the membership function μ_B attains all the values between 0 and 1.

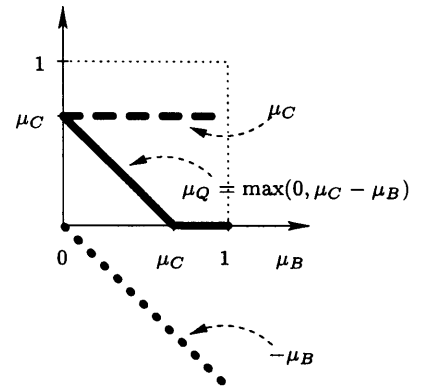**Solution:** The membership function of the composition is given by

$$\mu_{P \circ Q}(x, z) = \underset{y}{\mathsf{s}}\big(\mathsf{t}\big(\mu_P(x, y), \mu_Q(y, z)\big)\big) = \underset{y}{\sup}\big(\inf\big(\mu_P(x, y), \mu_Q(y, z)\big)\big)$$

$$= \underset{y}{\sup}\Big(\inf\Big(\min\big(1, \mu_A(x) + \mu_B(y)\big), \max\big(0, \mu_C(z) - \mu_B(y)\big)\Big)\Big),$$

where s and t are the s-function and the t-function, respectively. To determine $\mu_{P \circ Q}$, we first determine μ_P and μ_Q.

$$\mu_P(x, y) = \min\big(1, \mu_A(x) + \mu_B(y)\big).$$
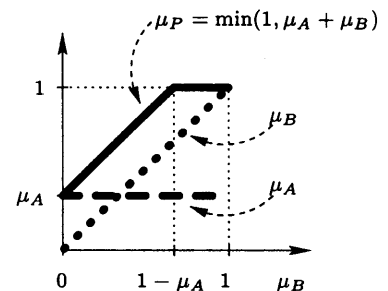
$$\mu_Q(y, z) = \max\big(0, \mu_C(z) - \mu_B(y)\big).$$

In order to determine the $\inf(\mu_P, \mu_Q)$, we need to plot the two solid lines in the above figures and choose the minimum. However, the minimum will depend on the relationship between μ_A and μ_C. So, we need to consider two cases.
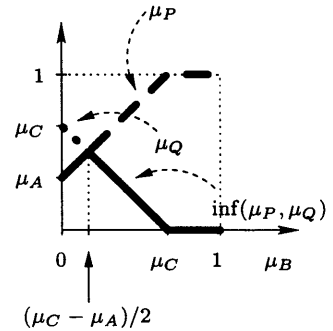
$\mu_A \leq \mu_C$

In this case, the two solid lines in the above figures intersect at a point such that

$$\mu_A + \mu_B = \mu_C - \mu_B,$$

or at $\mu_B = (\mu_C - \mu_A)/2$. As we can observe from the figure, the maximum value of the minimum curve with respect to μ_B is attained at $\mu_B = (\mu_C - \mu_A)/2$, and this maximum value is

$$\mu_{P \circ Q} = \sup_y \big(\inf\big(\min(1, \mu_A + \mu_B), \max(0, \mu_C - \mu_B)\big)\big)$$
$$= [\mu_A + \mu_B]_{\mu_B = (\mu_C - \mu_A)/2}$$
$$= [\mu_C - \mu_B]_{\mu_B = (\mu_C - \mu_A)/2}$$
$$= \frac{\mu_A + \mu_C}{2}.$$

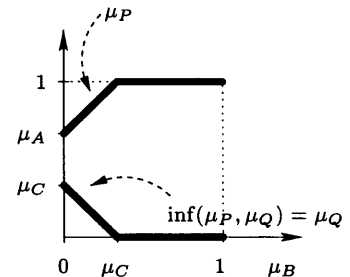## $\mu_A > \mu_C$

In this case, the two solid lines in the above figures do not intersect. As a result,

$$\mu_{P \circ Q} = \sup_y \big(\inf\big(\min(1, \mu_A + \mu_B), \max(0, \mu_C - \mu_B)\big)\big)$$
$$= \max(0, \mu_C - \mu_B);$$

and the maximum value is attained at the boundary when $\mu_B = 0$, where its value is μ_C.

Combining the two cases, we get

$$\mu_{P \circ Q}(x, z) = \begin{cases} \big(\mu_A(x) + \mu_C(z)\big)/2, & \text{if } \mu_A(x) \leq \mu_C(z); \\ \mu_C(z), & \text{if } \mu_A(x) > \mu_C(z); \end{cases}$$

or more compactly

$$\mu_{P \circ Q}(x, z) = \big(\min\big(\mu_A(x), \mu_C(z)\big) + \mu_C(z)\big)/2.$$
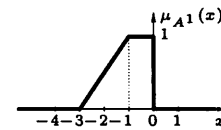
4. Consider a triple-rule fuzzy logic system, such that

$$\mathcal{R}^1: \quad A^1 \longrightarrow B^1,$$
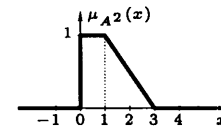$$\mathcal{R}^2: \quad A^2 \longrightarrow B^2,$$
$$\mathcal{R}^3: \quad A^3 \longrightarrow B^3,$$

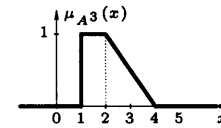where A^1, A^2, A^3, B^1, B^2, and B^3 are fuzzy sets with the membership functions

$$\mu_{A^1}(x) = \begin{cases} (x+3)/2, & \text{if } -3 < x \leq -1; \\ 1, & \text{if } -1 < x \leq 0; \\ 0, & \text{otherwise}; \end{cases}$$
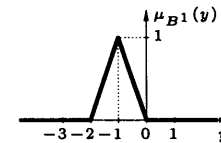
$$\mu_{A^2}(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ -(x-3)/2, & \text{if } 1 < x \leq 3; \\ 0, & \text{otherwise}; \end{cases}$$
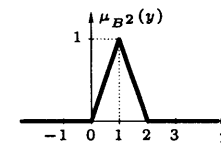
$$\mu_{A^3}(x) = \begin{cases} 1, & \text{if } 1 \leq x \leq 2; \\ -(x-4)/2, & \text{if } 2 < x \leq 4; \\ 0, & \text{otherwise}; \end{cases}$$

$$\mu_{B^1}(y) = \begin{cases} 1 - |y+1|, & \text{if } |y+1| \leq 1; \\ 0, & \text{otherwise}; \end{cases}$$
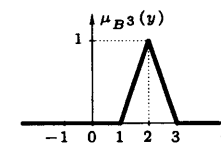
$$\mu_{B^2}(y) = \begin{cases} 1 - |y-1|, & \text{if } |y-1| \leq 1; \\ 0, & \text{otherwise}; \end{cases}$$
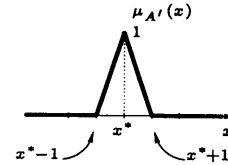
and

$$\mu_{B^3}(y) = \begin{cases} 1 - |y-2|, & \text{if } |y-2| \leq 1; \\ 0, & \text{otherwise}; \end{cases}$$

respectively. Assume the input value is $x^* = 0$. Determine the corresponding output y^* by assuming the minimum inference-engine with triangular fuzzifier such that

$$\mu_{A'}(x) = \begin{cases} 1 - |x - x^*|, & \text{if } |x - x^*| \leq 1; \\ 0, & \text{otherwise}; \end{cases}$$

and center-average defuzzifier. In other words, assume the triangular fuzzifier, the individual-based inference, the Mamdani-minimum implication, the supremum as the s-function, the infimum as the t-function, and the center-average defuzzifier. Show all your work.

Solution: The minimum inference-engine provides an output fuzzy set B' with the membership function

$$\mu_{B'}(y) = \sup_l \left(\sup_{x_1,\ldots,x_n} \left(\min\left(\mu_{A_1'}(x_1),\ldots,\mu_{A_n'}(x_n),\mu_{A_1^l}(x_1),\ldots,\mu_{A_n^l}(x_n),\mu_{B^l}(y) \right) \right) \mu_{B^l}(y) \right).$$
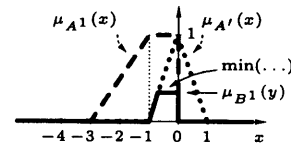
In our case, the input is one dimensional, and there are three rules, so

$$\mu_{B'^l}(y) = \sup_x \left(\min\left(\mu_{A'}(x),\mu_{A^l}(x),\mu_{B^l}(y) \right) \right),$$

for rules $\mathcal{R}^l$, $l = 1, 2$, and 3.

The expression $\min\left(\mu_{A'}(x),\mu_{A^1}(x),\mu_{B^1}(y) \right)$ for $x^* = 0$
gives

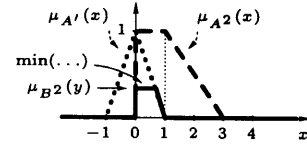$$\min\left(\mu_{A'}(x),\mu_{A^1}(x),\mu_{B^1}(y) \right)$$

$$= \begin{cases} \min\left(1 + x, \mu_{B^1}(y)\right), & \text{if } -1 < x \leq 0; \\ 0, & \text{otherwise}; \end{cases}$$

$$= \begin{cases} 1 + x, & \text{if } -1 < x \leq \mu_{B^1}(y) - 1; \\ \mu_{B^1}(y), & \text{if } \mu_{B^1}(y) - 1 < x \leq 0; \\ 0, & \text{otherwise}. \end{cases}$$

So, $\mu_{B'^1}(y) = \sup_x \left(\min\left(\mu_{A'}(x),\mu_{A^1}(x),\mu_{B^1}(y) \right) \right) = \mu_{B^1}(y)$.

The expression $\min\big(\mu_{A'}(x), \mu_{A^2}(x), \mu_{B^2}(y)\big)$ for $x^* = 0$ gives

$$\min\big(\mu_{A'}(x), \mu_{A^2}(x), \mu_{B^2}(y)\big)$$

$$= \begin{cases} \min\big(1 - x, \mu_{B^2}(y)\big), & \text{if } 0 < x \le 1; \\ 0, & \text{otherwise}; \end{cases}$$

$$= \begin{cases} \mu_{B^2}(y), & \text{if } 0 < x \le 1 - \mu_{B^2}(y); \\ 1 - x, & \text{if } 1 - \mu_{B^2}(y) < x \le 1; \\ 0, & \text{otherwise}. \end{cases}$$

So, $\mu_{B'^2}(y) = \sup_x \big(\min\big(\mu_{A'}(x), \mu_{A^2}(x), \mu_{B^2}(y)\big)\big) = \mu_{B^2}(y)$.

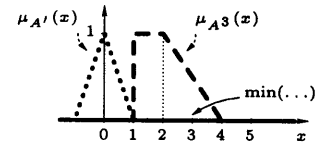The expression $\min\big(\mu_{A'}(x), \mu_{A^3}(x), \mu_{B^3}(y)\big)$ for $x^* = 0$ gives

$$\min\big(\mu_{A'}(x), \mu_{A^3}(x), \mu_{B^3}(y)\big) = 0.$$

So, $\mu_{B'^3}(y) = \sup_x \big(\min\big(\mu_{A'}(x), \mu_{A^3}(x), \mu_{B^3}(y)\big)\big) = 0$.

The center-average defuzzifier gives

$$y^* = \frac{\sum_l \bar{y}^l \, \mathrm{hgt}\big(B'^l\big)}{\sum_l \mathrm{hgt}\big(B'^l\big)},$$

where $\bar{y}^l$ is the center (of gravity) of the fuzzy set B'^l, and $\mathrm{hgt}\,(\cdot)$ is the height of a fuzzy set. In our case, $\bar{y}^1 = -1$, $\mathrm{hgt}\big(B'^1\big) = 1$, $\bar{y}^2 = 1$, $\mathrm{hgt}\big(B'^2\big) = 1$, and $\mathrm{hgt}\big(B'^3\big) = 0$. Therefore,

$$y^* = \frac{(-1)(1) + (1)(1) + (\bar{y}^3)(0)}{(1) + (1) + (0)},$$

or

$$y^* = 0.$$