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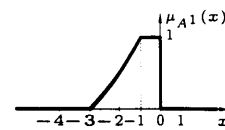
1. Consider a double-rule fuzzy logic system, such that

$$\mathcal{R}^1: A^1 \longrightarrow B^1,$$

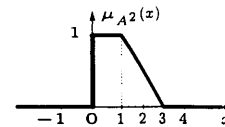
$$\mathcal{R}^2: A^2 \longrightarrow B^2,$$

where A^1 , A^2 , B^1 , and B^2 are fuzzy sets with the membership functions

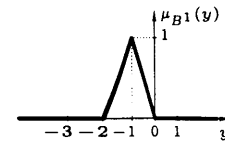
$$\mu_{A^1}(x) = \begin{cases} (x+3)/2, & \text{if } -3 < x \leq -1; \\ 1, & \text{if } -1 < x \leq 0; \\ 0, & \text{otherwise;} \end{cases}$$



$$\mu_{A^2}(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ -(x-3)/2, & \text{if } 1 < x \leq 3; \\ 0, & \text{otherwise;} \end{cases}$$

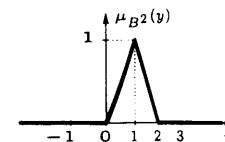


$$\mu_{B^1}(y) = \begin{cases} 1 - |y+1|, & \text{if } |y+1| \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$



and

$$\mu_{B^2}(y) = \begin{cases} 1 - |y-1|, & \text{if } |y-1| \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$



respectively.

- Obtain the compact formula $f_{\text{product}}(x)$ for the fuzzy system by assuming the product-inference engine with singleton fuzzifier and maximum defuzzifier. (30pts)
- Obtain the compact formula $f_{\text{minimum}}(x)$ for the fuzzy system by assuming the minimum-inference engine with singleton fuzzifier and maximum defuzzifier. (10pts)

2. Consider the continuous function

$$g(x) = 1 - x^2$$

that is defined on $x \in [-1, 1]$.

- (a) Design a fuzzy logic system with minimum number of triangular membership functions and the center-average defuzzifier to uniformly approximate g with a guaranteed accuracy such that

$$\sup_x |g(x) - f(x)| < 0.5,$$

where $f(x)$ is the output of the fuzzy system for the input x . Clearly describe the designed fuzzy sets and the rules. State the types of the fuzzifier and the inference engine to be used. (25pts)

- (b) Determine the theoretically possible maximum error, if the maximum defuzzifier is used in the fuzzy-logic system of the previous part instead of the center-average defuzzifier. (10pts)

3. The following input-output data have been provided for a nonlinear function $g(x)$ for $x \in [0, 1]$.

x	0	0.10	0.25	0.50	1
$g(x)$	2	1.98	1.88	1.50	0

Design a fuzzy logic system approximating $g(x)$ for $x \in [0, 1]$ based on the given points using a table look-up approach with triangular membership functions. (25pts)

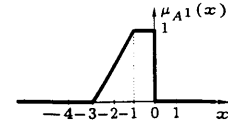
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1. Consider a double-rule fuzzy logic system, such that

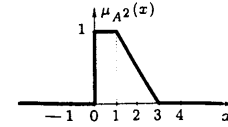
$$\begin{aligned}\mathcal{R}^1 &: A^1 \longrightarrow B^1, \\ \mathcal{R}^2 &: A^2 \longrightarrow B^2,\end{aligned}$$

where A^1 , A^2 , B^1 , and B^2 are fuzzy sets with the membership functions

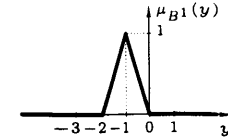
$$\mu_{A^1}(x) = \begin{cases} (x+3)/2, & \text{if } -3 < x \leq -1; \\ 1, & \text{if } -1 < x \leq 0; \\ 0, & \text{otherwise;} \end{cases}$$



$$\mu_{A^2}(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1; \\ -(x-3)/2, & \text{if } 1 < x \leq 3; \\ 0, & \text{otherwise;} \end{cases}$$

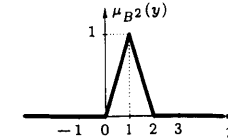


$$\mu_{B^1}(y) = \begin{cases} 1 - |y+1|, & \text{if } |y+1| \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$



and

$$\mu_{B^2}(y) = \begin{cases} 1 - |y-1|, & \text{if } |y-1| \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$



respectively.

- (a) Obtain the compact formula $f_{\text{product}}(x)$ for the fuzzy system by assuming the product-inference engine with singleton fuzzifier and maximum defuzzifier.

Solution: For the product-inference engine with singleton fuzzifier and maximum defuzzifier, the compact formula

$$f_{\text{product}}(x) = \bar{y}^{l^*},$$

where l^* is from the rule \mathcal{R}^{l^*} , such that

$$\mu_{A_1^{l^*}}(x) \mu_{A_2^{l^*}}(x) \cdots \mu_{A_n^{l^*}}(x) \geq \mu_{A_1^l}(x) \mu_{A_2^l}(x) \cdots \mu_{A_n^l}(x)$$

for all l , and \bar{y}^{l*} is the center (of gravity) of the fuzzy set B^{l*} . In our case, the input is one dimensional, and there are two rules, so

$$f_{\text{product}}(x) = \begin{cases} \bar{y}^1, & \text{if } \mu_{A^1}(x) \geq \mu_{A^2}(x); \\ \bar{y}^2, & \text{otherwise.} \end{cases}$$

From the given functions μ_{B^1} and μ_{B^2} , we observe that $\bar{y}^1 = -1$ and $\bar{y}^2 = 1$, respectively. Moreover, from the given functions μ_{A^1} and μ_{A^2} , we observe that when $x \leq 0$, we have $\mu_{A^1}(x) \geq \mu_{A^2}(x)$; since $\mu_{A^1}(x) \geq 0$, and $\mu_{A^2}(x) = 0$ for $x \leq 0$. Therefore,

$$f_{\text{product}}(x) = \begin{cases} -1, & \text{if } x \leq 0; \\ 1, & \text{otherwise.} \end{cases}$$

Note that when $\mu_{A^1}(x) = \mu_{A^2}(x)$, the choice for l^* is arbitrary. Here, we made one particular choice in the final expression of f_{product} , but other choices are also possible.

- (b) Obtain the compact formula $f_{\text{minimum}}(x)$ for the fuzzy system by assuming the minimum-inference engine with singleton fuzzifier and maximum defuzzifier.

Solution: For the minimum-inference engine with singleton fuzzifier and maximum defuzzifier, the compact formula

$$f_{\text{minimum}}(x) = \bar{y}^{l^*},$$

where l^* is from the rule \mathcal{R}^{l^*} , such that

$$\min(\mu_{A_1^{l^*}}(x), \mu_{A_2^{l^*}}(x), \dots, \mu_{A_n^{l^*}}(x)) \geq \min((\mu_{A_1^l}(x), \mu_{A_2^l}(x), \dots, \mu_{A_n^l}(x)))$$

for all l , and \bar{y}^{l*} is the center (of gravity) of the fuzzy set B^{l*} . In our case, the input is one dimensional and there are two rules, so

$$f_{\text{minimum}}(x) = \begin{cases} \bar{y}^1, & \text{if } \mu_{A^1}(x) \geq \mu_{A^2}(x); \\ \bar{y}^2, & \text{otherwise.} \end{cases}$$

From the given functions μ_{B^1} and μ_{B^2} , we observe that $\bar{y}^1 = -1$ and $\bar{y}^2 = 1$, respectively. Moreover, from the given functions μ_{A^1} and μ_{A^2} , we observe that when $x \leq 0$, we have $\mu_{A^1}(x) \geq \mu_{A^2}(x)$; since $\mu_{A^1}(x) \geq 0$, and $\mu_{A^2}(x) = 0$ for $x \leq 0$. Therefore,

$$f_{\text{minimum}}(x) = \begin{cases} -1, & \text{if } x \leq 0; \\ 1, & \text{otherwise.} \end{cases}$$

Note that when $\mu_{A^1}(x) = \mu_{A^2}(x)$, the choice for l^* is arbitrary. Here, we made one particular choice in the final expression of f_{minimum} , but other choices are also possible.

2. Consider the continuous function

$$g(x) = 1 - x^2$$

that is defined on $x \in [-1, 1]$.

- (a) Design a fuzzy logic system with minimum number of triangular membership functions and the center-average defuzzifier to uniformly approximate g with a guaranteed accuracy such that

$$\sup_x |g(x) - f(x)| < 0.5,$$

where $f(x)$ is the output of the fuzzy system for the input x . Clearly describe the designed fuzzy sets and the rules. State the types of the fuzzifier and the inference engine to be used.

Solution: A fuzzy logic system described by the function f with normal, consistent, and complete triangular membership functions with the center-average defuzzifier approximates any twice-differentiable function g , such that

$$\sup_x |g(x) - f(x)| \leq \frac{1}{8} \left(\sup_x \left| \frac{d^2 g(x)}{dx^2} \right| \right) h^2,$$

where h is the maximum separation between adjacent triangular membership functions. In our case,

$$\frac{d^2 g(x)}{dx^2} = \frac{d^2(1-x^2)}{dx^2} = \frac{d(-2x)}{dx} = -2.$$

From the desired accuracy requirement and from the above error expression, we need to have

$$\frac{1}{8} \left(\sup_x \left| \frac{d^2 g(x)}{dx^2} \right| \right) h^2 < 0.5$$

$$\frac{1}{8} \left(\sup_{x \in [-1,1]} |-2| \right) h^2 < 0.5$$

$$\frac{h^2}{4} < 0.5,$$

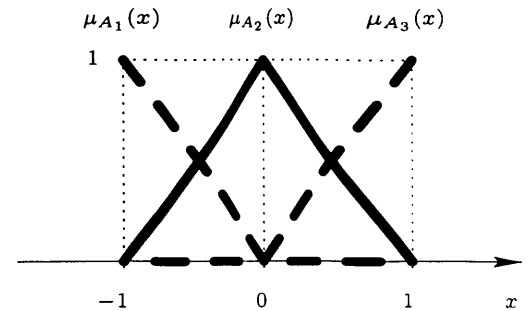
or $h < \sqrt{2} = 1.41$. Since we need a membership function at the two ends of the region $[-1, 1]$, we need at least one other membership function to have $h < 1.41$. Let the other one be centered at 0, so that $h = 1$. Therefore, the input fuzzy membership functions are

$$\mu_{A_1}(x) = \begin{cases} -x, & \text{if } -1 \leq x \leq 0; \\ 0, & \text{otherwise;} \end{cases}$$

$$\mu_{A_2}(x) = 1 - |x|,$$

and

$$\mu_{A_3}(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$



To design for the output sets, we need to determine the values of g at the centers of the triangular input sets.

$$g(\bar{x}_1) = g(-1) = 0,$$

$$g(\bar{x}_2) = g(0) = 1,$$

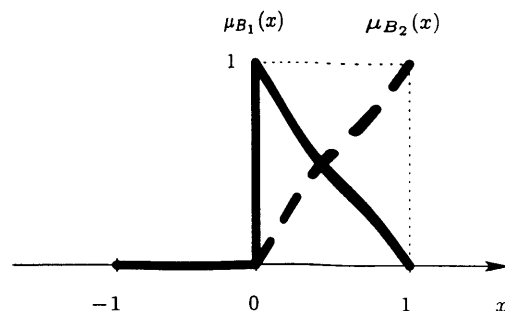
$$g(\bar{x}_3) = g(1) = 0,$$

where \bar{x}_i is the center of the fuzzy set A_i for $i = 1, 2, 3$. Since there are two distinct values at the output, we need at least two output membership functions. The only requirements we have on these output membership functions are that they are normal and their centers are at 0 and 1. One choice for the output membership functions is

$$\mu_{B_1}(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{otherwise;} \end{cases}$$

and

$$\mu_{B_2}(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$



With these input and output fuzzy sets, the rules of the fuzzy system are

$$\mathcal{R}^1: A_1 \longrightarrow B_1,$$

$$\mathcal{R}^2: A_2 \longrightarrow B_2,$$

$$\mathcal{R}^3: A_3 \longrightarrow B_1.$$

- (b) Determine the theoretically possible maximum error, if the maximum defuzzifier is used in the fuzzy-logic system of the previous part instead of the center-average defuzzifier.

Solution: A fuzzy logic system described by the function f with normal, consistent, and complete triangular membership functions with the maximum defuzzifier approximates any differentiable function g , such that

$$\sup_x |g(x) - f(x)| \leq \left(\sup_x \left| \frac{dg(x)}{dx} \right| \right) h,$$

where h is the maximum separation between adjacent triangular membership functions. In our case,

$$\sup_x |g(x) - f(x)| \leq \left(\sup_{x \in [-1, 1]} |-2x| \right) (1) = 2.$$

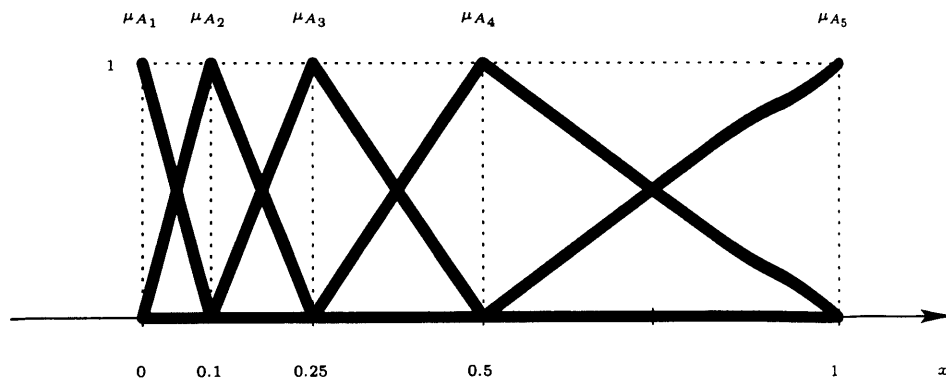
Therefore, the theoretically possible maximum error, $\sup_x |g(x) - f(x)| \leq 2$.

3. The following input-output data have been provided for a nonlinear function $g(x)$ for $x \in [0, 1]$.

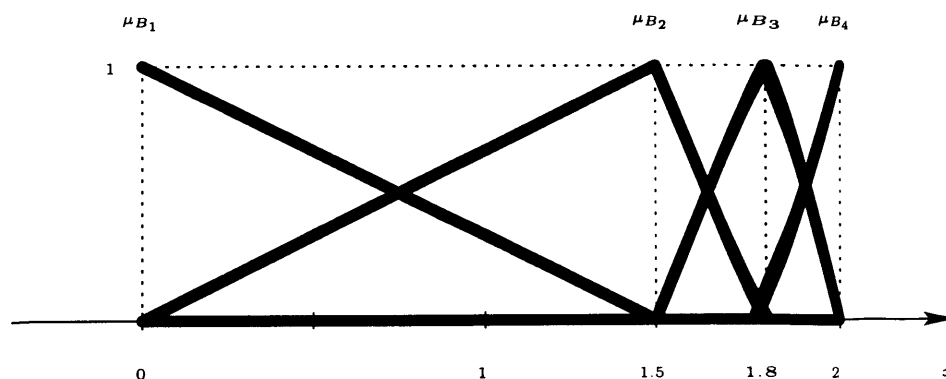
x	0	0.10	0.25	0.50	1
$g(x)$	2	1.98	1.88	1.50	0

Design a fuzzy logic system approximating $g(x)$ for $x \in [0, 1]$ based on the given points using a table look-up approach with triangular membership functions.

Solution: The crucial factor in the choice of the input membership functions is consistency. For a normal and complete set of membership functions, we should not obtain conflicting rules based on the given data. One way to avoid such conflict is to associate a different membership function for each input data point. Therefore, one possible set of input membership functions is given in the following figure.



There is no consistency problem with the output membership functions, but we may want to separate the output data points on different membership function to preserve the resolution of the given function. One possible set of output membership functions is given in the following figure.



With these input and output fuzzy sets, the rules of the fuzzy system are

$$\mathcal{R}^1: A_1 \longrightarrow B_4,$$

$$\mathcal{R}^2: A_2 \longrightarrow B_4,$$

$$\mathcal{R}^3: A_3 \longrightarrow B_3,$$

$$\mathcal{R}^4: A_4 \longrightarrow B_2,$$

$$\mathcal{R}^5: A_5 \longrightarrow B_1.$$