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1. The dynamics of a control system are described by the equations

$$\ddot{y}_1(t) + 8\dot{y}_1(t) + 11y_1(t) + 2\dot{y}_2(t) + 2y_2(t) = \dot{u}_1(t) + u_2(t),$$

$$y_1(t) - \dot{y}_2(t) = -\dot{u}_1(t) - u_2(t),$$

where u_1 and u_2 are the input and y_1 and y_2 are the output variables. Obtain a minimal state-space representation. Show all your work. (30pts)

2. A control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system. (30pts)

3. A continuous-time linear control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t),$$

where u , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively. Design an output feedback controller for the system, such that the 2% settling time is about 2 seconds, and the maximum percent-overshoot for a step-input is about 20%. (25pts)

HINT: The 2% settling time $t_{2\%s} = (4/\sigma_o)$, and the maximum percent-overshoot

$$M_{p\%} = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} 100\%$$

for a second-order system with no zero and the poles at $-\sigma_o \pm j\omega_d = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n$.

4. A linear control system is described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{u}(t),$$

$$\mathbf{y}(t) = \begin{bmatrix} -2 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 & 1 \end{bmatrix} \mathbf{u}(t),$$

where \mathbf{u} , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively.

- (a) Determine all the eigenvalues of the system. (05pts)
- (b) Determine all the zeros of the system. (10pts)

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1. The dynamics of a control system are described by the equations

$$\begin{aligned}\ddot{y}_1(t) + 8\dot{y}_1(t) + 11y_1(t) + 2\dot{y}_2(t) + 2y_2(t) &= \dot{u}_1(t) + u_2(t), \\ y_1(t) - \dot{y}_2(t) &= -\dot{u}_1(t) - u_2(t),\end{aligned}$$

where u_1 and u_2 are the input and y_1 and y_2 are the output variables. Obtain a minimal state-space representation. Show all your work.

Solution: In order to have a state-space representation, we may obtain the transfer matrix and generate a coprime factorization. To obtain the transfer matrix, we take the laplace transformation of the system equations under zero-initial conditions.

$$\begin{aligned}(s^2 + 8s + 11)Y_1(s) + (2s + 2)Y_2(s) &= sU_1(s) + U_2(s), \\ Y_1(s) - sY_2(s) &= -sU_1(s) - U_2(s);\end{aligned}$$

$$\begin{bmatrix} s^2 + 8s + 11 & 2s + 2 \\ 1 & -s \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s & 1 \\ -s & -1 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix};$$

or

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} s^2 + 8s + 11 & 2s + 2 \\ 1 & -s \end{bmatrix}^{-1} \begin{bmatrix} s & 1 \\ -s & -1 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix},$$

where U_1 , U_2 , Y_1 , and Y_2 are the laplace transforms of u_1 , u_2 , y_1 , and y_2 , respectively. The above equation is in the left factorization form, where $\mathbf{Y}(s) = D_0^{-1}(s)N_0^{-1}(s)\mathbf{U}(s)$, $\mathbf{Y} = [Y_1 \ Y_2]^T$, and $\mathbf{U} = [U_1 \ U_2]^T$. To obtain a left coprime factorization, we form an augmented matrix from N_0 and D_0 , and perform row operations until we obtain a reduced form. In our case, the augmented matrix is

$$\left[N_0(s) \mid D_0(s) \right] = \left[\begin{array}{cc|cc} s & 1 & s^2 + 8s + 11 & 2s + 2 \\ -s & -1 & 1 & -s \end{array} \right].$$

Adding the second row to the first row, we get

$$\left[N_1(s) \mid D_1(s) \right] = \left[\begin{array}{cc|cc} 0 & 0 & s^2 + 8s + 12 & s + 2 \\ -s & -1 & 1 & -s \end{array} \right].$$

Dividing the first row by $s + 2$, we get

$$\left[N_2(s) \mid D_2(s) \right] = \left[\begin{array}{cc|cc} 0 & 0 & s + 6 & 1 \\ -s & -1 & 1 & -s \end{array} \right].$$

The last operation resulted in a coprime factorization, since the rank of the above augmented matrix will not drop for any value of s , and the degree of the determinant of $D(s)$ is the same as the sum of its highest row degrees. As a result, all we need to do is to realize the left coprime factorization. First, we decompose $D(s)$ and $N(s)$, such that

$$D(s) = S_r(s)D_{hr} + \Psi_r(s)D_{lr},$$

and

$$N(s) = S_r(s)N_{hr} + \Psi_r(s)N_{lr},$$

where

$$S_r(s) = \left[\begin{array}{c|c|c} s^{l_1} & & \\ \hline & s^{l_2} & \\ \hline & & \ddots \end{array} \right] \text{ and } \Psi_r(s) = \left[\begin{array}{c|c|c} s^{l_1-1} & \dots & 1 \\ \hline & s^{l_2-1} & \dots & 1 \\ \hline & & & \ddots \end{array} \right]$$

are block-diagonal matrices, and l_i is the highest degree of the polynomials on the i th row of $D(s)$. In our case, $l_1 = 1$, $l_2 = 1$,

$$S_r(s) = \left[\begin{array}{c|c} s & 0 \\ \hline 0 & s \end{array} \right], \text{ and } \Psi_r(s) = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right].$$

The decompositions become

$$D(s) = \left[\begin{array}{cc} s+6 & 1 \\ 1 & -s \end{array} \right] = S_r(s)D_{hr} + \Psi_r(s)D_{lr} = \left[\begin{array}{c|c} s & 0 \\ \hline 0 & s \end{array} \right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array} \right] + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right] \left[\begin{array}{c|c} 6 & 1 \\ \hline 1 & 0 \end{array} \right],$$

and

$$N(s) = \left[\begin{array}{cc} 0 & 0 \\ -s & -1 \end{array} \right] = S_r(s)N_{hr} + \Psi_r(s)N_{lr} = \left[\begin{array}{c|c} s & 0 \\ \hline 0 & s \end{array} \right] \left[\begin{array}{c|c} 0 & 0 \\ \hline -1 & 0 \end{array} \right] + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right] \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -1 \end{array} \right].$$

The controller canonical-form realization is, then, given by

$$\dot{\mathbf{x}}(t) = (A_o^0 - B_o^0 D_{lr} D_{hr}^{-1} C_o^0) \mathbf{x}(t) + (B_o^0 N_{lr} - B_o^0 D_{lr} D_{hr}^{-1} N_{hr}) \mathbf{u}(t)$$

$$\mathbf{y}(t) = (D_{hr}^{-1} C_o^0) \mathbf{x}(t) + (D_{hr}^{-1} N_{hr}) \mathbf{u}(t),$$

where

$$A_o^0 = \left[\begin{array}{c|c|c} \begin{array}{cccc} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{array} & \begin{array}{cccc} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{array} & \\ \hline & & \ddots & \end{array} \right],$$

B_o^0 is the identity matrix with dimension $\sum_i l_i$, and

$$C_o^0 = \left[\begin{array}{ccc|ccc} 1 & 0 & \dots & 0 & & \\ \hline & & & & 1 & 0 & \dots & 0 \\ & & & & \hline & & & & & \ddots & \end{array} \right].$$

In our case,

$$A_o^0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array} \right],$$

B_o^0 is the 2 dimensional identity matrix, and

$$C_o^0 = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right].$$

So,

$$\begin{aligned} A_o &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 1 \\ -1 & 0 \end{bmatrix}, \\ B_o &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ C_o &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned}$$

and

$$D_o = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Therefore, one possible state-space representation of the system is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -6 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{u}(t), \\ \mathbf{y}(t) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{u}(t), \end{aligned}$$

where \mathbf{u} , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively.

2. A control system is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t), \\ \mathbf{y}(t) &= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \mathbf{x}(t), \end{aligned}$$

where u , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively. Obtain its kalman decomposition that separates the controllable, uncontrollable, observable, and unobservable portions. Clearly mark the portions on the decomposed system.

Solution: The kalman decomposition will transform the system, such that

$$\begin{bmatrix} \dot{x}_{c,o}(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},o}(t) \\ \dot{x}_{\bar{c},\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} A_{c,o} & 0 & * & 0 \\ * & A_{c,\bar{o}} & * & * \\ 0 & 0 & A_{\bar{c},o} & 0 \\ 0 & 0 & * & A_{\bar{c},\bar{o}} \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} b_{c,o} \\ b_{c,\bar{o}} \\ 0 \\ 0 \end{bmatrix} u(t),$$

and

$$y(t) = \begin{bmatrix} c_{c,o} & 0 & c_{\bar{o},c} & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},o}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + Du(t),$$

where the controllable, uncontrollable, observable, and unobservable portions are denoted by the subscripts c, \bar{c} , o, and \bar{o} , respectively. From the form of the given system, we can observe that the controllable and uncontrollable portions are already separated. In an n th order system that is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t),$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t),$$

where \mathbf{u} , \mathbf{x} , and \mathbf{y} are the input, the state, and the output variables, respectively; the observability matrix for $n = 3$ is given by

$$\mathcal{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 4 & 4 \end{bmatrix}.$$

To separate the observable and the unobservable portions, we need to pick the linearly independent row vectors from the observability matrix. Since there is only one linearly independent row vector, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ in the observability matrix, the rest of the rows need to be supplied with other vectors that are linearly independent to the original vector in the transformation matrix. So, we let

$$S^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the new system matrices are

$$\bar{A} = S^T A (S^T)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\bar{B} = S^T B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

and

$$\bar{C} = C (S^T)^{-1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

After marking the state variables, we get the kalman decomposition.

$$\begin{aligned} \begin{bmatrix} \dot{x}_{c,o}(t) \\ \dot{x}_{c,\bar{o}}(t) \\ \dot{x}_{\bar{c},\bar{o}}(t) \end{bmatrix} &= \begin{bmatrix} A_{c,o} & 0 & 0 \\ * & A_{c,\bar{o}} & * \\ 0 & 0 & A_{\bar{c},\bar{o}} \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} B_{c,o} \\ B_{c,\bar{o}} \\ 0 \end{bmatrix} u(t) \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t), \end{aligned}$$

and

$$y(t) = \begin{bmatrix} C_{c,o} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{c,o}(t) \\ x_{c,\bar{o}}(t) \\ x_{\bar{c},\bar{o}}(t) \end{bmatrix}.$$

3. A continuous-time linear control system is described by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x}(t), \end{aligned}$$

where u , \mathbf{x} , and y are the input, the state, and the output variables, respectively. Design an output feedback controller for the system, such that the 2% settling time is about 2 seconds, and the maximum percent-overshoot for a step-input is about 20%.

HINT: The 2% settling time $t_{2\%s} = (4/\sigma_o)$, and the maximum percent-overshoot

$$M_{p\%} = e^{-\left(\zeta/\sqrt{1-\zeta^2}\right)\pi} 100\%$$

for a second-order system with no zero and the poles at $-\sigma_o \pm j\omega_d = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\omega_n$.

Solution: We determine the desired system closed-loop poles from the system requirements.

Given Requirements	General System Restrictions	Specific System Restrictions
Maximum percent overshoot for the unit-step input	$M_p \approx 20\%,$ or $e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2.$	$\zeta \approx \frac{ \ln(0.2) }{\sqrt{(\ln(0.2))^2 + (\pi)^2}},$ or $\zeta \approx 0.46;$ since $M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$, and $\zeta = \ln(M_p) /\sqrt{(\ln(M_p))^2 + (\pi)^2}.$
2% settling-time for the unit-step input	$t_{2\%s} \approx 2s,$ or $\frac{4}{\sigma_o} \approx 2.$	$\sigma_o \approx 2,$ since $t_{2\%s} = 4/\sigma_o.$

From the given requirements, we obtain

$$\omega_n = \frac{\sigma_o}{\zeta} \approx \frac{2}{0.46} = 4.39.$$

And, the desired closed-loop pole locations are at $p_{d1,2} = -\sigma_o \pm j\sqrt{1-\zeta^2}\omega_n = -2 \pm j3.9$. The desired characteristic polynomial q_{cd} can be obtained from the desired-pole locations, where

$$q_{cd}(s) = (s - (-2 + j3.9))(s - (-2 - j3.9)) = s^2 + 4s + 19.21.$$

The characteristic polynomial q_c under state-feedback gain $K = [k_1 \ k_2]$, such that the input $u = Kx$, can be determined from

$$\begin{aligned}
q_c(s) &= \det(sI - (A + BK)) \\
&= \det\left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \ k_2]\right)\right) \\
&= s^2 + (-k_2 + 2)s + (-k_1 - 1).
\end{aligned}$$

Setting $q_c(s) = q_{cd}(s)$, we get

$$-k_1 - 1 = 19.21,$$

or $k_1 = -20.21$; and

$$-k_2 + 2 = 4,$$

or $k_2 = -2$. Therefore,

$$K = [-20.21 \quad -2].$$

However, since only the output, not the state variable, is available, we need to design an observer and use the observer state variable \hat{x} instead of the state variable x .

The desired observer-characteristic polynomial q_{od} can be obtained from the desired observer-pole locations. Since there is no explicit specifications, we may choose the two desired observer-pole

locations ourselves. Choosing both poles at -10 that is faster than the system poles, we get the desired observer characteristic polynomial

$$q_{od}(s) = (s + 10)(s + 10) = s^2 + 20s + 100.$$

The observer-characteristic polynomial q_o under the observer-error gain $L = [l_1 \ l_2]^T$ can be determined from

$$\begin{aligned} q_o(s) &= \det(sI - (A + LC)) \\ &= \det\left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}\right)\right) \\ &= s^2 + (-l_1 - l_2 + 2)s + (-3l_1 - l_2 - 1). \end{aligned}$$

Setting $q_o(s) = q_{od}(s)$, we get

$$-3l_1 - l_2 - 1 = 100,$$

and

$$-l_1 - l_2 + 2 = 20.$$

Solving the two equations for l_1 and l_2 , we get

$$L = \begin{bmatrix} -41.5 \\ 23.5 \end{bmatrix}.$$

Therefore,

$$u(t) = \begin{bmatrix} -20.21 & -2 \end{bmatrix} \hat{x}(t) \text{ for } t \geq 0,$$

where

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + \begin{bmatrix} -41.5 \\ 23.5 \end{bmatrix} (C\hat{x}(t) - y(t)),$$

and A , B , and C are the state, the input, and the output matrices of the system, respectively.

4. A linear control system is described by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} -2 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 1 \end{bmatrix} u(t),$$

where u , x , and y are the input, the state, and the output variables, respectively.

(a) Determine all the eigenvalues of the system.

Solution: The eigenvalues of a linear control system described by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t),$$

is determined from the characteristic equation

$$\det(\lambda I - A) = 0.$$

In our case,

$$\det \begin{bmatrix} \lambda & -1 \\ -2 & \lambda - 1 \end{bmatrix} = \lambda(\lambda - 1) - 2 = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2) = 0.$$

So, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.

(b) Determine all the zeros of the system.

Solution: The zeros of a linear control system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + D\mathbf{u}(t),$$

is determined from the values of s that drop the rank of the matrix

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}.$$

In our case,

$$\text{rank} \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = \text{rank} \left[\begin{array}{cc|cc} s & -1 & -1 & 1 \\ -2 & s-1 & -1 & 1 \\ \hline -2 & -1 & -1 & 1 \end{array} \right] = \text{rank} \begin{bmatrix} s & -1 & -1 \\ -2 & s-1 & -1 \\ -2 & -1 & -1 \end{bmatrix},$$

since the last two columns of the 3×4 matrix are linearly dependent. All the values of s that make the determinant of the 3×3 matrix zero are the zeros of the system.

$$\det \begin{bmatrix} s & -1 & -1 \\ -2 & s-1 & -1 \\ -2 & -1 & -1 \end{bmatrix} = -s^2 - 2s = -s(s+2).$$

So, the zeros are $s_1 = -2$ and $s_2 = 0$.