

Exam #1
In-Class Portion
90 minutes

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- Determine the extremals of the functional

$$J(y) = \int_{x_0}^{x_f} \frac{\sqrt{1+y'^2}}{x} dx$$

in terms of two constants, that may be determined by the end-conditions.

- Find the extremals of the functional

$$J(y) = \int_{-1}^1 \left(\frac{\mu}{2} y''^2 + \rho y \right) dx$$

where μ and ρ are positive constants, and $y(-1) = y(1) = y'(-1) = y'(1) = 0$.

- Find the curve $y = y(x)$ which yields an extremum for the functional

$$J(y) = \int_0^2 \left((1/2)y'^2 + yy' + y' + 2y + x^2 \right) dx$$

where $y(0)$ and $y(2)$ are arbitrary.

- Find the minimizing extremals for the functional

$$J(y) = \int_{x_0}^{x_f} \sqrt{2+y'^2} dx$$

where (x_0, y_0) lies on the curve $y = x^2$, and (x_f, y_f) lies on the curve $y = x - 1$.

- Find all the broken extremals for the functional

$$J(y) = \int_0^2 \left(y'^4 - 6y'^2 \right) dx$$

having exactly one corner, and such that $y(0) = 0$, and $y(2) = 0$. Give explicit functions for each portion of the extremal.

- Find the optimal input $u(t)$ for $t \geq 0$, such that the cost function

$$J(x, u) = \frac{1}{2} \int_0^1 (x^2 + u^2) dt$$

is minimized; and $x(0) = 10$, $x(1) = 0$, and x satisfies $\dot{x}(t) = u(t)$.

#1

$$J(y) = \int_{x_0}^{x_f} \frac{\sqrt{1+y'^2}}{x} dx$$

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E. L. EQU. $f_y - \frac{d}{dx} f_{y'} = 0$

$$0 - \frac{d}{dx} \left(\frac{y'}{x \sqrt{1+y'^2}} \right) = 0$$

$$\frac{y'}{x \sqrt{1+y'^2}} = c_1$$

$$y'^2 = c_1^2 x^2 (1+y'^2)$$

$$y'^2 = \frac{c_1^2 x^2}{1 - c_1^2 x^2}$$

$$y' = \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}}$$

$$\int dy = \int \frac{c_1 x}{\sqrt{1 - c_1^2 x^2}} dx + c_2$$

$$y = \frac{1}{c_1} \sqrt{1 - c_1^2 x^2} + c_2 ; \text{ if } c_1 \neq 0$$

$$c_1^2 (y + c_3)^2 = 1 - c_1^2 x^2$$

$$x^2 + (y + c_3)^2 = \frac{1}{c_1^2}$$

$$x^2 + (y + y_c)^2 = r^2 \quad \begin{cases} (c_1 \neq 0 \text{ case}) \\ \text{a circle} \end{cases}$$

or

$$y = y_k \quad (c_1 = 0 \text{ case})$$

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#2

$$\mathcal{J}(y) = \int_1^l \left(\frac{\mu}{2} y''^2 + py \right) dx \quad \mu, p > 0$$

$$y^{(-1)} = y^{(1)} = y'^{(-1)} = y'^{(1)} = 0$$

$$\text{E.L. EQU} \quad F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$$

$$p - \frac{d}{dx}(0) + \frac{d^2}{dx^2}(\mu y'') = 0$$

$$\frac{d^2}{dx^2} y'' = -\frac{p}{\mu} \quad ; \quad y'' = -\frac{p}{\mu}$$

$$y = -\frac{1}{4 \cdot 3 \cdot 2} \frac{p}{\mu} x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0$$

$$\text{END COND. } y^{(-1)} = 0 \Rightarrow -\frac{1}{6} \frac{p}{\mu} (-1)^3 + 3c_3(-1)^2 + 2c_2(-1) + c_1 = 0$$

$$y^{(1)} = 0 \Rightarrow -\frac{1}{6} \frac{p}{\mu} (1)^3 + 3c_3(1)^2 + 2c_2(1) + c_1 = 0$$

Adding the two eqns.

$$6c_3 + 2c_1 = 0 ,$$

Subt. 1st from the 2nd

$$-\frac{1}{3} \frac{p}{\mu} + 4c_2 = 0$$

$$\text{so } c_1 = -3c_3$$

$$\text{and } c_2 = +\frac{1}{12} \frac{p}{\mu}$$

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$$y(-1) = 0 \Rightarrow -\frac{1}{24} \frac{f}{\mu} (-1)^4 + c_3(-1)^3 + c_2(-1)^2 + c_1(-1) + c_0 = 0$$

$$y(1) = 0 \Rightarrow -\frac{1}{24} \frac{f}{\mu} (1)^4 + c_3(1)^3 + c_2(1)^2 + c_1(1) + c_0 = 0$$

Adding the two eqns. $-\frac{1}{12} \frac{f}{\mu} + 2c_2 + 2c_0 = 0$

Subt. 1st from the 2nd $2c_3 + 2c_1 = 0$

$\left. \begin{matrix} \\ \\ \text{but since } c_1 = -3c_3 \end{matrix} \right\} \Rightarrow c_1 = c_3 = 0$

and $c_0 = \frac{1}{24} \frac{f}{\mu} - c_2 = \frac{1}{24} \frac{f}{\mu} - \frac{1}{12} \frac{f}{\mu}$

$$= -\frac{1}{24} \frac{f}{\mu}$$

i.e. $y(x) = -\frac{1}{24} \frac{f}{\mu} x^4 + \frac{1}{12} \frac{f}{\mu} x^2 - \frac{1}{24} \frac{f}{\mu}$

$$= -\frac{1}{24} \frac{f}{\mu} (x^4 - 2x^2 + 1)$$

$$= -\frac{1}{24} \frac{f}{\mu} (x+1)^2 (x-1)^2$$

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#3

$$\int y = \int^2 \left(\frac{1}{2}y'^2 + yy' + y' + 2y + x^2 \right) dx$$

 $y(0)$ & $y(2)$ arbitrary

$$E-L.Eqn. \quad f_y - \frac{d}{dx} f_{y'} = 0$$

$$(y'+2) - \frac{d}{dx}(y'+y+1) = 0$$

$$2-y''=0$$

$$y''=2$$

$$y = x^2 + c_1 x + c_2$$

$$END. \text{ COND. } y(0) = \text{free} ; f_{y'} \Big|_{x=0} = 0$$

$$\left[y' + y + 1 \right]_{x=0} = 0$$

$$\left[(2x+c_1) + (x^2+c_1x+c_2) + 1 \right]_{x=0} = 0$$

$$c_1 + c_2 + 1 = 0 \quad (1)$$

$$y(2) = \text{free} ; f_{y'} \Big|_{x=2} = 0$$

$$\left[(2x+c_1) + (x^2+c_1x+c_2) + 1 \right]_{x=2} = 0$$

$$3c_1 + c_2 + 9 = 0 \quad (2)$$

Subtract (1) from (2) $\Rightarrow c_1 = -4 \Rightarrow c_2 = 3$

$$\text{so } y(x) = x^2 - 4x + 3 \\ = (x-1)(x-3)$$

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#4

$$J(y) = \int_{x_0}^{x_f} \sqrt{2+y'^2} dx \quad \begin{aligned} & (x_0, y_0) \text{ on } y=x^2 \\ & (x_f, y_f) \text{ on } y=x-1 \end{aligned}$$

E.L.EQN. $F - y' f y' = c$, since F doesn't explicitly depend on x .

$$\sqrt{2+y'^2} - y' \left(\frac{y'}{\sqrt{2+y'^2}} \right) = c,$$

$$\frac{2+y'^2-y'^2}{\sqrt{2+y'^2}} = c, \quad ; \quad 2 = c, \sqrt{2+y'^2}$$

$$2+y'^2 = c_2 \quad ; \quad y' = c_3$$

$$\text{so } y(x) = c_3 x + c_4$$

END cond. (x_0, y_0) on $y=x^2$

$$\Rightarrow \left[F + \left(\frac{d}{dx}(x^2) - y' \right) f y' \right] = 0$$

$$\left[\sqrt{2+y'^2} + (2x-y') \frac{y'}{\sqrt{2+y'^2}} \right] = 0$$

$$\left[2+2xy' \right]_{\substack{x=x_0 \\ y=y_0}} = 0$$

$$\text{since } y' = c_3 \quad ; \quad 1+x_0 c_3 = 0$$

$$\Rightarrow y_0 = x_0^2$$

$$c_3 x_0 + c_4 = x_0^2 \quad ; \quad c_4 = x_0^2 + 1$$

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 (x_f, y_f) on $y = x - 1$

$$\Rightarrow \left[f + \left(\frac{d}{dx}(x-1) - y' \right) f_y' \right]_{\substack{x=x_f \\ y=y_f}} = 0$$

$$\left[\sqrt{2+y'^2} + (1-y') \frac{y'}{\sqrt{2+y'^2}} \right]_{\substack{x=x_f \\ y=y_f}} = 0$$

$$\left[2+y' \right]_{\substack{x=x_f \\ y=y_f}} = 0$$

$$2+c_3=0 \quad ; \quad c_3=-2$$

$$\Rightarrow x_0 = -\frac{1}{c_3} = \frac{1}{2}$$

$$\Rightarrow c_4 = x_0^2 + 1 = \frac{5}{4}$$

$$\Rightarrow y_f = x_f - 1$$

$$\Rightarrow y_0 = x_0^2 = \frac{1}{4}$$

$$-2x_f + \frac{5}{4} = x_f - 1 \Rightarrow x_f = \frac{3}{4}$$

$$\Rightarrow y_f = x_f - 1 = -\frac{1}{4}$$

i.e. $y(x) = -2x + \frac{5}{4} \quad ; \quad (x_0, y_0) = \left(\frac{1}{2}, \frac{1}{4} \right)$

$$(x_f, y_f) = \left(\frac{3}{4}, -\frac{1}{4} \right)$$

#5

$$\int_0^2 (y'^4 - 6y'^2) dx \quad y(0)=0 \\ \text{ONE-CORNER} \quad y'(2)=0$$

E.L.-EQN. $F - y' F y' = c$, since F does not explicitly depend on x .

$$(y'^4 - 6y'^2) - y' (4y'^3 - 12y') = 0$$

\hookrightarrow poly. in y' , if there is a soln. then $y' = c_2$ is the soln.

$$y' = c_2 ; y = c_2 x + c_3$$

END COND. $y(0)=0$, let $y_1(x)$ be the soln. in the 1st int.

$$y_1(0)=0 ; c_3=0$$

i.e. $y_1 = c_2 x$ in $[0, c]$

$y'(2)=0$, let $y_2(x)$ be the soln. in the 2nd. int.

$$y_2(2)=0 ; c_{2_2}^2 + c_{3_2} = 0 ; c_{3_2} = -2c_{2_2}$$

i.e. $y_2 = c_{2_2}(x-2)$ in $[c, 2]$

or let $y_1 = \alpha x$; $x \in [0, c]$

$$y_2 = \beta(x-2) ; x \in [c, 2]$$

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Corner Case. $f'y' = \text{cont. at } x=c$

$$\text{since } f'y' = 4y'^3 - 12y'$$

$$[4y'^3 - 12y' = 4y_2'^3 - 12y_2']_{x=c}$$

$$[4\alpha^3 - 12\alpha = 4\beta^3 - 12\beta]_{x=c}$$

$$4\alpha(\alpha^2 - 3) = 4\beta(\beta^2 - 3)$$

$$\text{soln. } \alpha, \beta \in \{0, \pm\sqrt{3}\}$$

 $f-y'f'y' = \text{cont. at } x=c$

$$\text{since } F-y'Fy' = -3y'^4 + 6y'^2$$

$$[-3y'^4 + 6y'^2 = -3y_2'^4 + 6y_2'^2]_{x=c}$$

$$[-3\alpha^4 - 6\alpha^2 = -3\beta^4 - 6\beta^2]_{x=c}$$

$$-3\alpha^2(\alpha^2 - 2) = -3\beta^2(\beta^2 - 2)$$

$$\text{soln. } \alpha, \beta \in \{0, \pm\sqrt{2}\} \text{ or } \alpha^2 = \beta^2$$

The only intersection of the two solns. is

if $\alpha, \beta \in \{\pm\sqrt{3}\}$ where $\alpha \neq \beta$ for corner

i.e. $\begin{cases} y_1 = \sqrt{3}x \\ y_2 = -\sqrt{3}(x-2) \end{cases}$ or $\begin{cases} y_1 = -\sqrt{3}x \\ y_2 = \sqrt{3}(x-2) \end{cases}$

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#6

$$J = \frac{1}{2} \int_0^1 (x^2 + u^2) dt$$

$$x(0) = 10$$

$$x(1) = 0$$

$$\text{and } \dot{x} = u$$

$$\text{Hamiltonian } H = \frac{1}{2} (x^2 + u^2) + \lambda u$$

OPT. EQU.

$$\dot{x} = H_x \implies \dot{x} = u$$

$$\dot{\lambda} = -H_u \implies \dot{\lambda} = -x \implies \ddot{\lambda} = -\dot{x} \implies \ddot{x} = x$$

$$\ddot{O} = H_u \implies u = -\lambda$$

$$\text{or } \ddot{x} - x = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - 1 \right) x = 0$$

$$\left(\frac{\partial}{\partial t} - 1 \right) \left(\frac{\partial}{\partial t} + 1 \right) x = 0$$

$$\text{i.e. } x(t) = c_1 e^t + c_2 e^{-t}$$

$$\text{END. (contd.) } x(0) = 10 \implies c_1 + c_2 = 10, \quad c_2 = 10 - c_1$$

$$\text{i.e. } x(t) = c_1 e^t + (10 - c_1) e^{-t}$$

$$x(1) = 0 \implies c_1 e + (10 - c_1) e^{-1} = 0$$

$$c_1 = \frac{10e^{-1}}{e^{-1} - e}$$

$$\text{i.e. } x(t) = \frac{10}{e^{-1} - e} e^{-1} e^t + \frac{10}{e^{-1} - e} e e^{-t}$$

$$= \frac{10}{e^{-1} - e} (e^{t-1} - e^{-(t-1)}) ; t \in [0, 1]$$

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since $\lambda = -\dot{x}$; $\lambda(t) = -\frac{10}{e^{-t} - e} (e^{t-1} + e^{-(t-1)})$

and since $u = -\lambda$; $u(t) = \frac{10}{e^{-t} - e} (e^{t-1} + e^{-(t-1)})$

or $u(t) = \frac{e^{t-1} + e^{-(t-1)}}{e^{t-1} - e^{-(t-1)}} x(t) ; t \in [0, 1]$