

Exam#1
In-Class Portion
75 minutes

Nov. 03, 1992

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1. Find the solution $x = x(t)$ for $0 \leq t \leq 2$ that yields an extremum for the cost function

$$J(x) = \int_0^2 \left(\frac{1}{2} (\dot{x}(t))^2 + x(t)\dot{x}(t) + \dot{x}(t) + 2x(t) + t^2 \right) dt,$$

where $x(0)$ and $x(2)$ are arbitrary.

2. Find all the broken extremals of the cost function (15pts)

$$J(x) = \int_0^2 \left((\dot{x}(t))^4 - 6(\dot{x}(t))^2 \right) dt,$$

having exactly one corner, and such that $x(0) = 0$, and $x(2) = 0$. Give explicit functions for each portion of the extremal.

3. Given the cost function (20pts)

$$J = \frac{1}{2} \int_0^1 \left((x(t))^2 + (u(t))^2 \right) dt,$$

and $x(0) = 10$, $x(1) = 0$, where x satisfies $\dot{x}(t) = u(t)$, determine the optimal input $u(t)$ for $0 \leq t \leq 1$ in feedback form by using

- (a) the optimality equations based on the Hamiltonian, and
 (b) the solution to the Riccati equation. (10pts)

(10pts)

4. Consider a system described by

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - u(t) \\ \dot{x}_2(t) &= -2x_2(t) - 2u(t).\end{aligned}$$

Assuming $|u| \leq 1$, determine the trajectory of the state vector $[x_1 \ x_2]^T$ under a control which will transfer the states from $[-5 \ 7]^T$ to $[-2 \ 0]^T$ in minimum time. Plot the trajectory, and determine all the state variables at the switching point(s), but do not compute the switching time instance(s). If there are more than one possible trajectories, show all of them. (25pts)

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1. Find the solution $x = x(t)$ for $0 \leq t \leq 2$ that yields an extremum for the cost function

$$J(x) = \int_0^2 \left(\frac{1}{2} (\dot{x}(t))^2 + x(t)\dot{x}(t) + \dot{x}(t) + 2x(t) + t^2 \right) dt,$$

where $x(0)$ and $x(2)$ are arbitrary.

Solution: Given the cost function

$$J = \int_0^2 \Phi(x, \dot{x}, t) dt = \int_0^2 \left(\frac{1}{2} (\dot{x}(t))^2 + x(t)\dot{x}(t) + \dot{x}(t) + 2x(t) + t^2 \right) dt,$$

where $x(0)$ and $x(2)$ are free. The Euler-Langrange equation gives

$$\Phi_x - \frac{d}{dt} \Phi_{\dot{x}} = 0,$$

$$(\dot{x} + 2) - \frac{d}{dt}(\dot{x} + x + 1) = 0,$$

$$\ddot{x} = 2,$$

or

$$x(t) = t^2 + c_1 t + c_2,$$

for some constants c_1 and c_2 . We need to determine the unknown constants from the boundary conditions. At $t = 0$, $x(0)$ is free, so

$$[\Phi_{\dot{x}}]_{t=0} = 0,$$

$$[\dot{x} + x + 1]_{t=0} = 0,$$

$$[(2t + c_1) + (t^2 + c_1 t + c_2) + 1]_{t=0} = 0,$$

or

$$c_1 + c_2 + 1 = 0. \quad (1.1)$$

At $t = 2$, $x(2)$ is free, so similarly

$$[\Phi_{\dot{x}}]_{t=2} = 0,$$

$$[(2t + c_1) + (t^2 + c_1 t + c_2) + 1]_{t=2} = 0,$$

or

$$3c_1 + c_2 + 9 = 0. \quad (1.2)$$

Subtracting (??) from (??), we obtain $c_1 = -4$, and $c_2 = 3$. Therefore, the optimal solution is

$$x(t) = t^2 - 4t + 3 \text{ for } 0 \leq t \leq 2.$$

#2

$$J(x) = \int_0^2 (\dot{x}^4 - 6\dot{x}^2) dt$$

$$\text{s.t. } x(0) = 0$$

$$x(2) = 0$$

For $\phi = \dot{x}^4 - 6\dot{x}^2$, the E-L equation is

$$\phi_x - \frac{d}{dt} \phi_{\dot{x}} = 0$$

or $0 - \frac{d}{dt} (4\dot{x}^3 - 12\dot{x}) = 0$

$$4\dot{x}^3 - 12\dot{x} = \text{constant}$$

\hookrightarrow real valued solution gives $\dot{x} = c$ (constant)

For solutions with corners, we need to satisfy
W-E corner conditions.

① $\phi_{\dot{x}}$ is continuous at the corner

$$\phi_{\dot{x}} = 4\dot{x}^3 - 12\dot{x}$$

$$\text{So } 4c_1^3 - 12c_1 = 4c_2^3 - 12c_2$$

where $\dot{x} = c_1$ for $t \in [0, t_1]$, $\dot{x} = c_2$ for $t \in (t_1, 2]$

(2) $\phi - \dot{x}\phi_{\dot{x}}$ is continuous at the corner

$$\begin{aligned}\phi - \dot{x}\phi_{\dot{x}} &= \dot{x}^4 - 6\dot{x}^2 - \dot{x}(4\dot{x}^3 - 12\dot{x}) \\ &= -3\dot{x}^2(\dot{x}^2 - 2)\end{aligned}$$

$$\text{So } -3c_1^2(c_1^2 - 2) = -3c_2^2(c_2^2 - 2)$$

Since c_1 and c_2 are independent with the exception that $c_1 \neq c_2$, let

$$c_1^2(c_1^2 - 2) = K_1$$

$$c_1^4 - 2c_1^2 - K_1 = 0$$

$$c_1^2 = -1 \pm \sqrt{1+K_1}$$

$$\text{for real } c_1, \quad c_1^2 = -1 + \sqrt{1+K_1}$$

$$\text{and } 1+K_1 > 1 \text{ or } K_1 > 0$$

$$\text{or } c_1^2 = K_2 > 0$$

$$\text{or } c_1 = \pm \sqrt{K_2} = \pm K$$

Substituting $c_1 = K$ and $c_2 = -K$ so that $c_1 \neq c_2$ into the condition (1), we get

$$4K^3 - 12K = 4(-K)^3 - 12(-K)$$

$$8K^3 - 24K = 0, \quad 8K(K^2 - 3) = 0$$

$\Rightarrow K=0$ but $\pm K$ doesn't give different solutions

$$\Rightarrow K^2 - 3 = 0, \quad K = \pm\sqrt{3}$$

So, we get

(a) $\dot{x} = \sqrt{3}$ for $t \in [0, t_1]$
 $\dot{x} = -\sqrt{3}$ for $t \in (t_1, 2]$

(b) $\dot{x} = -\sqrt{3}$ for $t \in [0, t_1]$
 $\dot{x} = \sqrt{3}$ for $t \in (t_1, 2]$

In case (a), $t \in [0, t_1]$, $x(t) = \sqrt{3}t + c_3$

since $x(0) = 0$, $c_3 = 0$
or $x(t) = \sqrt{3}t$

$$t \in (t_1, 2], \quad x(t) = -\sqrt{3}t + c_4$$

since $x(2) = 0$, $c_4 = 2\sqrt{3}$
or $x(t) = -\sqrt{3}(t - 2)$

In case (B), similarly

$$t \in [0, t_1], \quad x(t) = -\sqrt{3}t$$

$$t \in [t_1, 2], \quad x(t) = \sqrt{3}(t-2)$$

So $x(t) = \begin{cases} \sqrt{3}t, & \text{if } 0 \leq t < t_1; \\ -\sqrt{3}(t-2), & \text{if } t_1 \leq t \leq 2. \end{cases}$

or

$$x(t) = \begin{cases} -\sqrt{3}t, & \text{if } 0 \leq t < t_1; \\ \sqrt{3}(t-2), & \text{if } t_1 < t \leq 2. \end{cases}$$

#3

$$\mathcal{J} = \frac{1}{2} \int_0^1 (x^2 + u^2) dt$$

where $x(0) = 10$ and $\dot{x} = u$
 $x(1) = 0$

④ Hamiltonian approach

$$\text{Let } H = \frac{1}{2} \dot{x}^2 + \frac{1}{2} \dot{u}^2 + \lambda u$$

The optimality equations are

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$$\dot{x} = H_\lambda \Rightarrow \dot{x} = u$$

$$\dot{\lambda} = -H_x \Rightarrow \dot{\lambda} = -x$$

$$0 = Hu \Rightarrow 0 = u + \lambda \Rightarrow u = -\lambda$$

or $\ddot{x} = u = -\lambda$

$$\ddot{x} = -\ddot{\lambda} = -(-x)$$

$$\ddot{x} = x$$

$$\ddot{x} - x = 0 \Rightarrow x(t) = K_1 e^t + K_2 e^{-t}$$

$$x(0) = 10 \Rightarrow 10 = K_1 + K_2$$

$$x(1) = 0 \Rightarrow 0 = K_1 e + K_2 e^{-1}$$

or $K_1 = -\frac{10}{e^2 - 1}$

$$K_2 = \frac{10e^2}{e^2 - 1}$$

so $x(t) = \frac{10}{e^2 - 1} (e^{-(t-2)} - e^t)$ for $t \in [0, 1]$

$$\dot{x}(t) = -\frac{10}{e^2 - 1} (e^{-(t-2)} - e^t)$$

$$\lambda(t) = -\frac{10}{e^2 - 1} (-e^{-(t-2)} - e^t)$$

So $u(t) = -\lambda(t)$

$$u(t) = -\frac{10}{e^2-1} (e^{-(t-2)} + e^t) \text{ for } t \in [0, 1].$$

(b) Riccati equation

$$\dot{P} = -A^T P - PA + PBR^{-1}B^TP - Q$$

$$\text{where } P(1) = S$$

since in this case $S = \infty$ or $S^\dagger = 0$

we need to use the inverse Riccati equation

$$\dot{P}^\dagger = P^\dagger A^T + AP^\dagger - BR^{-1}B^T + P^\dagger Q P^\dagger$$

$$\text{where } P = P^\dagger, \text{ let } q = P^\dagger$$

$$A = 0, B = 1$$

$$Q = 1, R = 1$$

So $\dot{q} = q(0) + (0)q - (1)(1)\tilde{(1)}(1) + q(1)q$

$$\dot{q} = -1 + q^2$$

$$\int \frac{dq}{q^2-1} = \int dt + c,$$

$$\int \left(\frac{y_2}{q-1} - \frac{y_2}{q+1} \right) dq = t + c,$$

$$\frac{1}{2} \ln(q-1) - \frac{1}{2} \ln(q+1) = t + c,$$

$$\ln\left(\frac{q-1}{q+1}\right) = 2t + 2c,$$

$$\frac{q-1}{q+1} = c_2 e^{2t}$$

$$\text{let } q(1)=0 \Rightarrow -1 = c_2 e^2 \text{ or } c_2 = -e^{-2}$$

$$\text{So } \frac{q-1}{q+1} = -e^{2t-2}$$

$$q-1 = -e^{2t-2}(q+1)$$

$$(1+e^{2t-2})q = 1-e^{2t-2}$$

$$q = \frac{1-e^{2t-2}}{1+e^{2t-2}} \text{ or } p = q^{-1} = \frac{1+e^{2t-2}}{1-e^{2t-2}}$$

$$\text{So } u(t) = -p x(t) = -\frac{1+e^{2(t-1)}}{1-e^{2(t-1)}} x(t) \text{ for } t \in [0, 1]$$

NOTE: "p" from part (b) matches $-\frac{u(t)}{x(t)}$ from part (a).

#4

$$\dot{x}_1 = -x_1 - u$$

$$\dot{x}_2 = -2x_2 - 2u$$

Min-time with

$$x_1(0) = -5$$

$$x_1(\tau) = -2$$

$$x_2(0) = 7$$

$$x_2(\tau) = 0$$

and $|u| \leq 1$

In this case $H = 1 + \lambda_1(-x_1 - u) + \lambda_2(-2x_2 - 2u)$
and the optimality conditions are

$$\ddot{x} = H_x \Rightarrow$$

$$\ddot{x}_1 = -x_1 - u$$

$$\ddot{x}_2 = -2x_2 - 2u$$

$$\ddot{\lambda} = -H_{\lambda} \Rightarrow$$

$$\ddot{\lambda}_1 = \lambda_1$$

$$\ddot{\lambda}_2 = 2\lambda_2$$

$$H^* \leq H / \begin{matrix} x=x^* \\ \lambda=\lambda^* \end{matrix}$$

$$\Rightarrow -\lambda_1^* u^* - 2\lambda_2^* u^* \leq -\lambda_1^* u - 2\lambda_2^* u$$

$$\text{or } -(\lambda_1^* + 2\lambda_2^*) u^* \leq -(\lambda_1^* + 2\lambda_2^*) u$$

$$\text{so } u^* = \text{sgn}(\lambda_1 + 2\lambda_2)$$

For $u=1$, $\dot{x}_1 = -x_1 - 1 \Rightarrow x_1 = e^{-t} x_{1(0)} - 1$
 $\dot{x}_2 = -2x_2 - 2 \Rightarrow x_2 = e^{-2t} x_{2(0)} - 1$

From the 1st equation $(x_1 + 1)^2 = e^{-2t} x_{1(0)}^2$

and the 2nd equation $x_2 + 1 = e^{-2t} x_{2(0)}$

or by dividing

$$\frac{(x_1 + 1)^2}{x_2 + 1} = c_1 \text{ (constant)}$$

or $x_2 = c_2 (x_1 + 1)^2 - 1$

A parabola

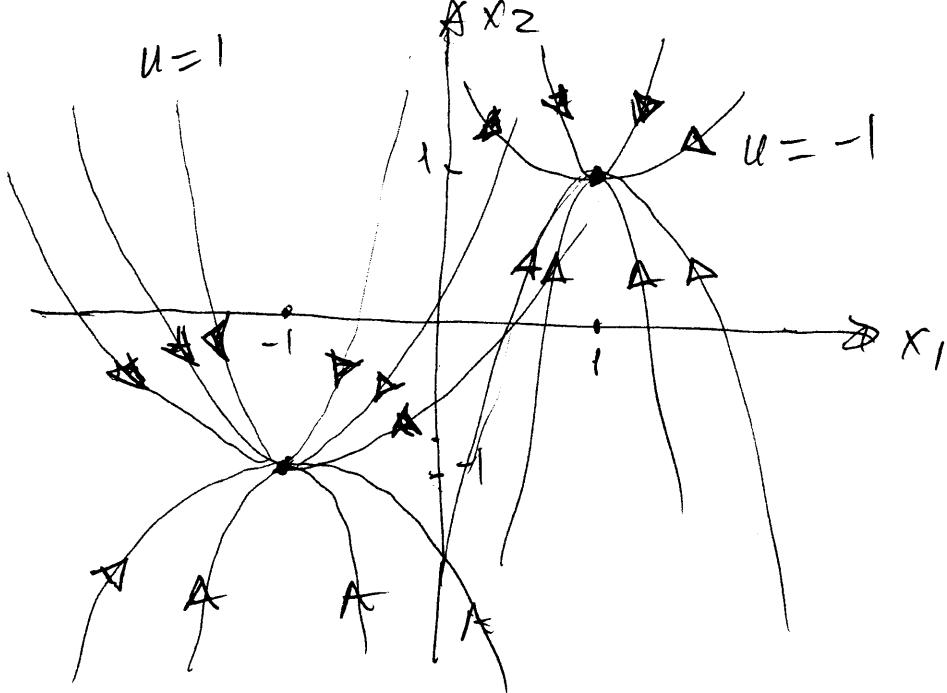
For $u=-1$, $\dot{x}_1 = -x_1 + 1 \Rightarrow x_1 = e^{-t} x_{1(0)} + 1$
 $\dot{x}_2 = -2x_2 + 2 \Rightarrow x_2 = e^{-2t} x_{2(0)} + 1$

Similberly

$$\frac{(x_1 - 1)^2}{x_2 - 1} = c_3$$

or $x_2 = c_4 (x_1 - 1)^2 + 1$

A also parabola



Since $\alpha - t \rightarrow \infty \Rightarrow x_i \rightarrow \pm 1$, all the trajectories will converge to the two vertices.

Since $x_1(\tau) = -2$
 $x_2(\tau) = 0$ we choose the two

that go through $(-2, 0)$.

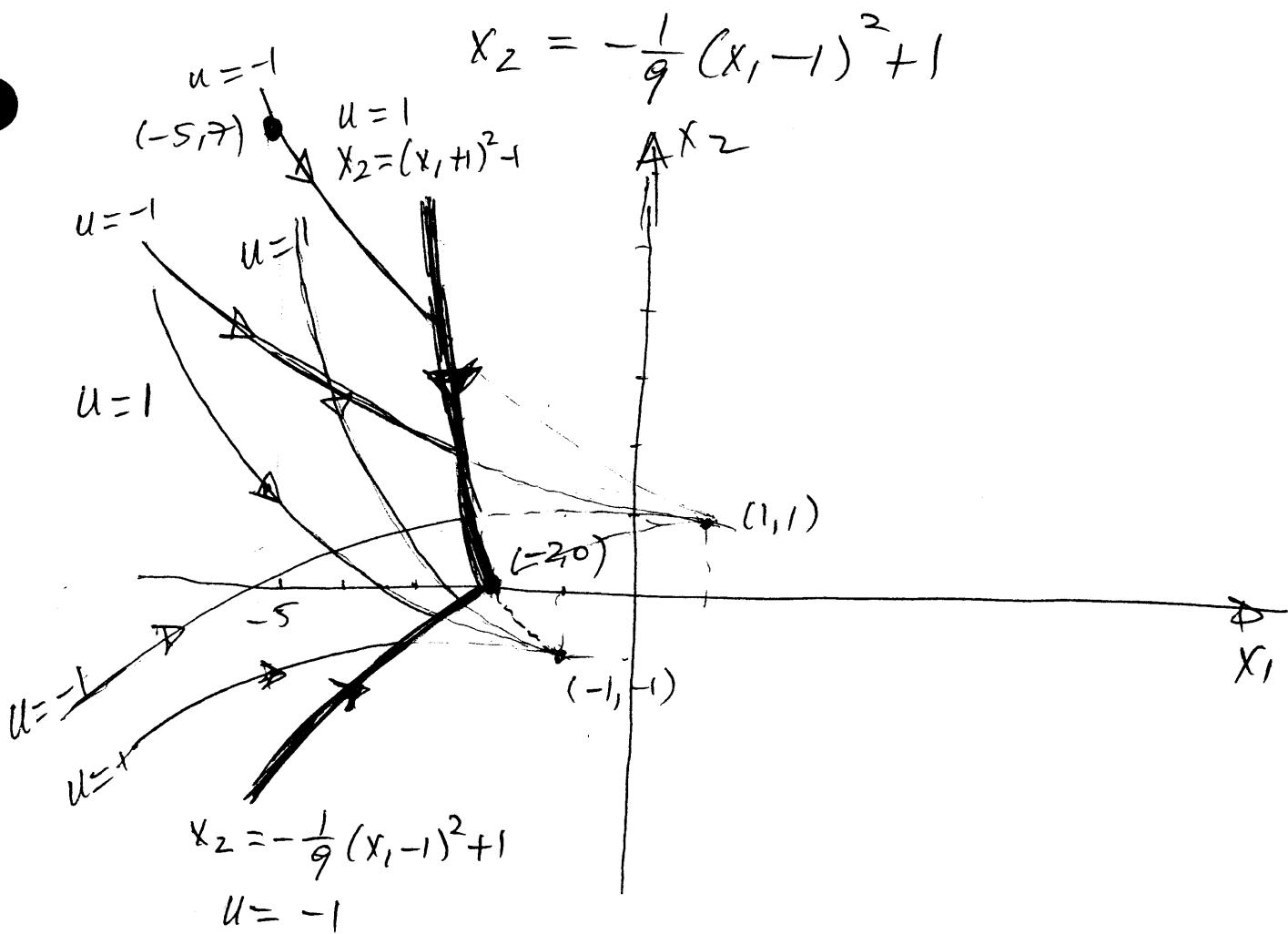
$$u = 1 \text{ case} \quad x_2 = c_2 (x_1 + 1)^2 - 1$$

$$0 = c_2 (-2 + 1)^2 - 1 \Rightarrow c_2 = 1$$

$$x_2 = (x_1 + 1)^2 - 1$$

$$u = -1 \text{ case} \quad x_2 = c_4 (x_1 - 1)^2 + 1$$

$$0 = c_4 (-2 - 1)^2 + 1 \Rightarrow c_4 = -\frac{1}{9}$$

CASE 1

$$u = -1 \text{ for } t \in [0, t_1)$$

$$\text{on } x_2 = c_4 (x_1 - 1)^2 + 1$$

$$\text{and at } t = 0, \quad 7 = c_4 (-5 - 1)^2 + 1 \\ \Rightarrow c_4 = \frac{1}{6}$$

$$\text{or on } x_2 = \frac{1}{6} (x_1 - 1)^2 + 1$$

$u = 1$ for $t \in [t_1, \tau]$

on $x_2 = (x_1 + 1)^2 - 1$ (The intersection point)
is at $(-3.2330, 3.9864)$

So

$$\begin{array}{c} \left. \begin{array}{l} x_1(0) = -5 \\ x_2(0) = 7 \end{array} \right\} \xrightarrow[2]{u = -1} \left. \begin{array}{l} x_1(t_1) \\ x_2(t_1) \end{array} \right\} \xrightarrow[3]{u = 1} \left. \begin{array}{l} x_1(\tau) = -2 \\ x_2(\tau) = 0 \end{array} \right\} \\ \text{circled: } x_2 = \frac{1}{8}(x_1 - 1)^2 + 1 \quad \text{circled: } x_2 = (x_1 + 1)^2 - 1 \end{array}$$

CASE 2

$u = 1$ for $t \in [0, t_1)$

on $x_2 = c_2(x_1 + 1)^2 - 1$

and at $t=0$, $7 = c_2(-5+1)^2 - 1$

$$\Rightarrow c_2 = \frac{3}{8}$$

$$\text{or on } x_2 = \frac{3}{8}(x_1 + 1)^2 - 1$$

$u = -1$ for $t \in (t_1, \tau]$

on $x_2 = -\frac{1}{9}(x_1 - 1)^2 + 1$ (The intersection pt.
is at $(-2.3892, -0.2763)$)

So

$$\begin{array}{c} \left. \begin{array}{l} x_1(0) = -5 \\ x_2(0) = 7 \end{array} \right\} \xrightarrow[3]{u = 1} \left. \begin{array}{l} x_1(t_1) \\ x_2(t_1) \end{array} \right\} \xrightarrow[3]{u = -1} \left. \begin{array}{l} x_1(\tau) = -2 \\ x_2(\tau) = 0 \end{array} \right\} \\ \text{circled: } x_2 = \frac{3}{8}(x_1 + 1)^2 - 1 \quad \text{circled: } x_2 = -\frac{1}{9}(x_1 - 1)^2 + 1 \end{array}$$