EE 432 **Exam#1** **Mar. 09, 2004**

75 minutes

1. Obtain the extremal of J and analyze the result.

   (a)
   $$J(x) = \int_0^1 x\dot{x}\, dt.$$
   (10pts)

   (b)
   $$J(x) = \int_0^1 tx\dot{x}\, dt.$$
   (10pts)

2. Find an extremal of the functional
   $$J(x) = \int_0^4 \left((\dot{x}-1)^2(\dot{x}+1)^2\right) dt$$
   with $x(0) = 0$, and $x(4) = 2$, that has just one corner. (30pts)

3. Consider the cost function
   $$J(x_1, x_2, u) = \int_0^\infty \left(20{x_1}^2 + 5{x_2}^2 + u^2\right) dt,$$
   and a continuous-time linear control-system described by
   $$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -4x_1(t) - 4x_2(t) + u(t),\end{aligned}$$
   where u is the control variable, and x_1 and x_2 are the state variables.

   (a) Obtain the optimal feedback control that minimizes the cost function J. (30pts)

   (b) Determine the optimal cost J^* for an arbitrary initial state. (10pts)

4. Find the minimum-time control that will transfer an arbitrary initial state to the origin for the control system described by
   $$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - u(t)\\ \dot{x}_2(t) &= -2x_2(t) - 2u(t),\end{aligned}$$
   where u is the control variable, and x_1 and x_2 are the state variables, provided that $|u(t)| \le 1$ for $t \ge 0$. (30pts)

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Solutions

1. Obtain the extremal of J and analyze the result.

(a)

$$J(x) = \int_0^1 x\dot{x}\,\mathrm{d}t.$$

Solution: Given the functional

$$J = \int_0^1 \Phi(x,\dot{x},t)\,\mathrm{d}t = \int_0^1 x\dot{x}\,\mathrm{d}t,$$

where the end conditions are unspecified. The Euler-Lagrange equation gives

$$\Phi_x - \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\dot{x}} = 0,$$

$$(\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}(x) = 0,$$

or

$$0 = 0.$$

In other words, the Euler-Lagrange equation is always satisfied; and as long as the end conditions are satisfied, all trajectories are extremal.

This result is also justified from a further analysis of J, where

$$J = \int_0^1 x\dot{x}\,\mathrm{d}t = \int_0^1 x\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_{x(0)}^{x(1)} x\,\mathrm{d}x = \left[x^2/2\right]_{x=x(0)}^{x=x(1)} = \left(x^2(1) - x^2(0)\right)/2,$$

which only depends on the end conditions.

(b)

$$J(x) = \int_0^1 tx\dot{x}\,\mathrm{d}t.$$

Solution: Given the functional

$$J = \int_0^1 \Phi(x,\dot{x},t)\,\mathrm{d}t = \int_0^1 tx\dot{x}\,\mathrm{d}t,$$

where the end conditions are unspecified. The Euler-Lagrange equation gives

$$\Phi_x - \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\dot{x}} = 0,$$

$$(t\dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t}(tx) = 0,$$

$$t\dot{x} - \left(x + t\dot{x}\right) = 0,$$

or

$$x = 0 \text{ for } 0 \le t \le 1.$$

In other words, the optimal trajectory is $x(t) = 0$ for $0 \le t \le 1$, if the end conditions are such that $x(0) = x(1) = 0$; otherwise there is no extremal.

2. Find an extremal of the functional

$$J(x) = \int_0^4 \left((\dot{x}-1)^2(\dot{x}+1)^2\right)\, \mathrm{d}t$$

with $x(0) = 0$, and $x(4) = 2$, that has just one corner.

Solution: Given the cost function

$$J = \int_0^2 \Phi(x, \dot{x}, t)\, \mathrm{d}t = \int_0^4 \left((\dot{x}-1)^2(\dot{x}+1)^2\right)\, \mathrm{d}t,$$

where $x(0) = 0$, and $x(4) = 2$. The Euler-Lagrange's equation gives

$$\begin{aligned} \Phi_x - \frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\dot{x}} &= 0, \\ (0) - \frac{\mathrm{d}}{\mathrm{d}t}\left(2(\dot{x}^2-1)2\dot{x}\right) &= 0, \\ 4\dot{x}\left(\dot{x}^2-1\right) &= 0, \\ \dot{x}\left(\dot{x}^2-1\right) &= c, \end{aligned}$$

or

$$\dot{x}^3 - \dot{x} - c = 0$$

for a constant c. Solving the above polynomial equation for $\dot{x}$, we get $\dot{x} = a$, or $x(t) = at + b$ for some constants a and b.

In order for the extremal to have a corner, we need to have

$$x(t) = \begin{cases} a_1 t + b_1, & \text{if } t \in [0, t_1]; \\ a_2 t + b_2, & \text{if } t \in [t_1, 4]; \end{cases}$$

where $a_1 \neq a_2$; since $a_1 = a_2$ implies $\dot{x}(t_{1-} = \dot{x}(t_1+$ and no corner. Substituting the end conditions, we get

$$\begin{aligned} x(0) = 0 &\Longrightarrow a_1(0) + b_1 = 0, \text{ or } b_1 = 0; \\ x(4) = 2 &\Longrightarrow a_2(4) + b_2 = 2, \text{ or } b_2 = -4a_2 + 2. \end{aligned}$$

As a result,

$$x(t) = \begin{cases} a_1 t, & \text{if } t \in [0, t_1]; \\ a_2(t-4) + 2, & \text{if } t \in [t_1, 4]. \end{cases}$$

Next, we need to consider the Weierstrass-Erdmann's corner conditions at $t = t_1$. The first corner condition is

$$[\Phi_{\dot{x}}]_{t=t_{1-}} = [\Phi_{\dot{x}}]_{t=t_{1+}};$$

and since

$$\Phi_{\dot{x}} = 4\dot{x}(\dot{x}^2 - 1),$$

we get

$$4a_1({a_1}^2 - 1) = 4a_2({a_2}^2 - 1). \tag{2.1}$$

The second corner condition is

$$[\Phi - \dot{x}\Phi_{\dot{x}}]_{t=t_{1-}} = [\Phi - \dot{x}\Phi_{\dot{x}}]_{t=t_{1+}},$$

and since

$$\Phi - \dot{x}\Phi_{\dot{x}} = (\dot{x}^2-1)^2 - \dot{x}4\dot{x}(\dot{x}^2-1) = -(\dot{x}^2-1)(3\dot{x}^2+1),$$

we get

$$-({a_1}^2-1)(3{a_1}^2+1) = -({a_2}^2-1)(3{a_2}^2+1). \tag{2.2}$$

On possible solution to Equations (2.1) and (2.2) is when each side is zero. When each side of the Equation (2.1) is zero, we get $a_1,\ a_2 \in \{0, \pm 1\}$. Similarly, when each side of the Equation (2.2) is zero, we get $a_1,\ a_2 \in \{\pm 1\}$. As a result, two possible choices that would satisfy both of the corner conditions are $a_1 = \pm 1$ and $a_2 = \mp 1$. In other words,

$$x(t) = \begin{cases} t, & \text{if } t \in [0, t_1]; \\ -t+6, & \text{if } t \in [t_1, 4]; \end{cases}$$

or

$$x(t) = \begin{cases} -t, & \text{if } t \in [0, t_1]; \\ t-2, & \text{if } t \in [t_1, 4]. \end{cases}$$

We can determine t_1 from the continuity of x at $t = t_1$. For the first case, we get $t_1 = -t_1 + 6$ or $t_1 = 3$; and for the second case, we get $-t_1 = t_1 - 2$ or $t_1 = 1$. Therefore, one possible optimal trajectory with one corner is

$$x(t) = \begin{cases} t, & \text{if } t \in [0, 3]; \\ -t+6, & \text{if } t \in [3, 4]. \end{cases}$$

Another possible trajectory is

$$x(t) = \begin{cases} -t, & \text{if } t \in [0, 1]; \\ t-2, & \text{if } t \in [1, 4]. \end{cases}$$

3. Consider the cost function

$$J(x_1, x_2, u) = \int_0^\infty \left(20{x_1}^2 + 5{x_2}^2 + u^2\right)\, dt,$$

and a continuous-time linear control-system described by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -4x_1(t) - 4x_2(t) + u(t), \end{aligned}$$

where u is the control variable, and x_1 and x_2 are the state variables.

(a) Obtain the optimal feedback control that minimizes the cost function J.

Solution: Since the infinite-time cost function is quadratic in the state and the input variables, the optimal control can be expressed in state-feedback form, such that

$$u(t) = -R^{-1}B^TPx(t),$$

where R is from the cost function

$$J = \int_0^\infty \frac{1}{2}\left(\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{u}^T(t)R\mathbf{u}(t)\right)\, dt,$$

and P is the solution to the algebraic riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

for the control system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t).$$

In our case, we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

and

$$J = \int_0^\infty \frac{1}{2} \left(\mathbf{x}^T(t) \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix} \mathbf{x}(t) + u(t)(2)u(t) \right) \mathrm{d}t.$$

In other words,

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix}, \text{ and } R = 2.$$

Since P is symmetric, let

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

Substituting all these matrices into the algebraic riccati equation, we get

$$A^T P + PA - PBR^{-1}B^T P + Q = 0,$$

$$\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$
$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (2)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$$
$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 40 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

We get

$$-4p_2 - 4p_2 - (1/2){p_2}^2 + 40 = 0,$$
$$-4p_3 + p_1 - 4p_2 - (1/2)p_2 p_3 + 0 = 0,$$

and

$$p_2 - 4p_3 + p_2 - 4p_3 - (1/2){p_3}^2 + 10 = 0$$

from the $(1,1)$, $(1,2)$ (or $(2,1)$), and $(2,2)$ terms of the matrix equation, respectively. From the equation in the $(1,1)$ term, we have

$${p_2}^2 + 16p_2 - 80 = 0,$$

or $p_2 = -(16/2) \pm \sqrt{(16/2)^2 - (1)(-80)} = -8 \pm 12.$

- Assuming $p_2 = -8 - 12 = -20$, the equation in the $(2,2)$ term gives
$$ {p_3}^2 + 16p_3 - 4p_2 - 20 = 0, $$
$$ {p_3}^2 + 16p_3 + 60 = 0, $$
or $p_3 = -(16/2) \pm \sqrt{(16/2)^2 - (1)(60)} = -8 \pm 2$. Since P needs to be non-negative definite implying that the diagonal elements of P have to be positive, neither $p_3 = -6$ nor $p_3 = -10$ is acceptable. In other words, $p_2 = -20$ assumption doesn't give a non-negative definite matrix.
- Assuming $p_2 = -8 + 12 = 4$, the equation in the $(2,2)$ term gives
$$ {p_3}^2 + 16p_3 - 4p_2 - 20 = 0, $$
$$ {p_3}^2 + 16p_3 - 36 = 0, $$
or $p_3 = -(16/2) \pm \sqrt{(16/2)^2 - (1)(-36)} = -8 \pm 10$. Since P needs to be non-negative definite implying that the diagonal elements of P have to be positive, $p_3 = -18$ is not acceptable, and $p_3 = 2$. Finally, from the equation in the $(1,2)$ term, we get
$$ p_1 = 4p_2 + (1/2)p_2p_3 + 4p_3 = 28. $$
As a result, we get
$$ P = \begin{bmatrix} 28 & 4 \\ 4 & 2 \end{bmatrix} $$
that is positive definite and unique.

Therefore, the optimal control is
$$ u(t) = -R^{-1}B^TP\mathbf{x}(t) = -(2)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 28 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{x}(t), $$
or
$$ u(t) = -\begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{x}(t) = -\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \text{for } t \geq 0. $$

(b) Determine the optimal cost J^* for an arbitrary initial state.

Solution: For the infinite-time quadratic cost function with the state-feedback control, the optimal cost
$$ J^* = \frac{1}{2}\mathbf{x}^T(0)P\mathbf{x}(0), $$
where $\mathbf{x}$ is the state variable, and P is the solution to the algebraic riccati equation. In our case,
$$ P = \begin{bmatrix} 28 & 4 \\ 4 & 2 \end{bmatrix}; $$
and for $\mathbf{x}(0) = \begin{bmatrix} x_1(0) & x_2(0) \end{bmatrix}^T$, the optimal cost
$$ J^* = 14{x_1}^2(0) + 4x_1(0)x_2(0) + {x_2}^2(0). $$

4. Find the minimum-time control that will transfer an arbitrary initial state to the origin for the control system described by

$$\dot{x}_1(t) = -x_1(t) - u(t)$$
$$\dot{x}_2(t) = -2x_2(t) - 2u(t),$$

where u is the control variable, and x_1 and x_2 are the state variables, provided that $|u(t)| \leq 1$ for $t \geq 0$.

Solution: The Hamiltonian for the minimum-time control and the given system is

$$H\big(t, \mathbf{u}, \mathbf{x}, \boldsymbol{\lambda}\big) = 1 + \boldsymbol{\lambda}^T\big(A\mathbf{x} + B\mathbf{u}\big) = 1 + \lambda_1(-x_1 - u) + \lambda_2(-2x_2 - 2u),$$

where $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}^T$ is the langrange multiplier, and the system is described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t).$$

Here, $\mathbf{u}$ and $\mathbf{x}$ are the input and the state variables, respectively. The optimality conditions in terms of the Hamiltonian are

$$\dot{\mathbf{x}} = H_{\boldsymbol{\lambda}}; \quad \begin{aligned} \dot{x}_1 &= -x_1 - u, \\ \dot{x}_2 &= -2x_2 - 2u; \end{aligned}$$

$$\dot{\boldsymbol{\lambda}} = -H_{\mathbf{x}}; \quad \begin{aligned} \dot{\lambda}_1 &= -(-\lambda_1) = \lambda, \\ \dot{\lambda}_2 &= -(-2\lambda_2) = 2\lambda_2; \end{aligned}$$

and

$$\Big[H\Big]_{\substack{\mathbf{u}=\mathbf{u}^* \\ \mathbf{x}=\mathbf{x}^* \\ \boldsymbol{\lambda}=\boldsymbol{\lambda}^*}} \leq \Big[H\Big]_{\substack{\mathbf{x}=\mathbf{x}^* \\ \boldsymbol{\lambda}=\boldsymbol{\lambda}^*}}; \quad \begin{aligned} &1 + \lambda_1^*(-x_1^* - u^*) + \lambda_2^*(-2x_2^* - 2u^*) \leq 1 + \lambda_1^*(-x_1^* - u) + \lambda_2^*(-2x_2^* - 2u), \\ &\text{or } -(\lambda_1^* + 2\lambda_2^*)u^* \leq -(\lambda_1^* + 2\lambda_2^*)u, \end{aligned}$$

where $(\cdot)^*$ designates the optimal values. From the last optimality condition, we get

$$u^* = \operatorname{sgn}(\lambda_1 + 2\lambda_2).$$

In order to determine the optimal trajectory, we need to analyze the response when $u = \pm 1$. For $u = \pm 1$, we get

$$\dot{x}_1(t) = -x_1 \mp 1$$
$$\dot{x}_2(t) = -2x_2(t) \mp 2,$$

or

$$x_1(t) = c_1 e^{-t} \mp 1$$
$$x_2(t) = c_2 e^{-2t} \mp 2,$$

for $t \geq 0$, and for some constants c_1 and c_2. To get a state trajectory, we eliminate the time variable by using the first equation, so that

$$e^{-t} = (x_1 \pm 1)/c_1,$$

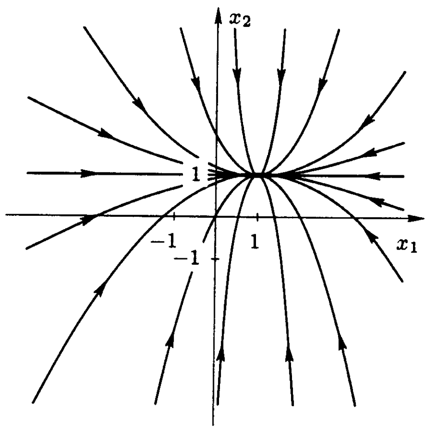
or

$$e^{-2t} = (x_1 \pm 1)^2/{c_1}^2.$$

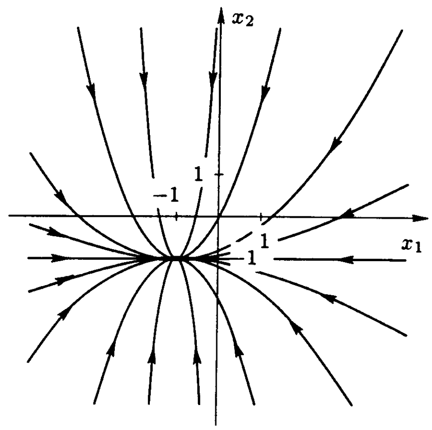
Substituting the above expression into the second equation, we get

$$x_2 = (c_2/c_1{}^2)(x_1 \pm 1)^2 \mp 1 = c_3(x_1 \pm 1)^2 \mp 1.$$

Therefore, the state trajectories are parabolas with vertexes at $(\pm 1, \pm 1)$; and since x_1, $x_2 \to \mp 1$ as $t \to \infty$, the direction of flow is toward these vertexes as shown in the following figures.



(a) $u = -1$ case.



(b) $u = 1$ case.

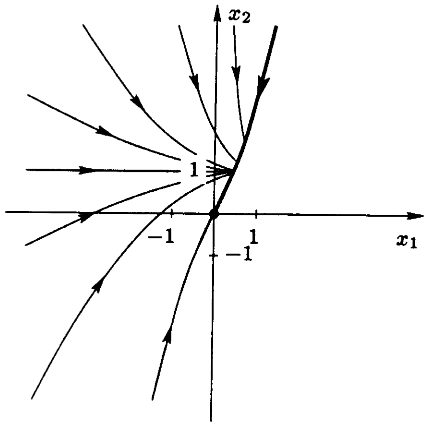
Since our destination is the origin, the last switch is to be to the curves that go through the origin, specifically

$$x_2 = -\,(x_1 - 1)^2 + 1, \text{ when } u = -1;$$

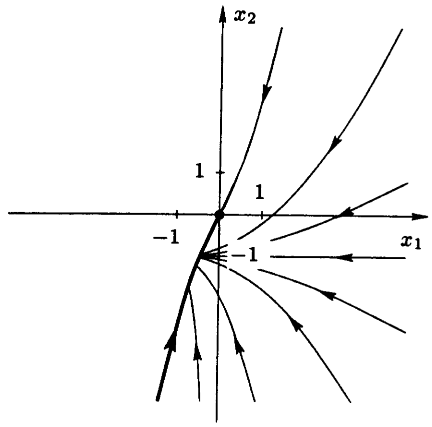
and

$$x_2 = (x_1 + 1)^2 - 1, \text{ when } u = 1.$$

To determine the control signal for each region, we choose the trajectories that intersect the above curves with different values of u as shown in the following figures. The first figure shows the region in the state trajectory, where the optimal control starts with $u = -1$; and when $x_2 = (x_1 + 1)^2 - 1$, that is shown by the thicker line in the first figure, the control is switched to $u = 1$. The second figure shows the region, where the optimal control starts with $u = 1$; and when $x_2 = -(x_1 - 1)^2 + 1$, that is shown by the thicker line in the second figure, the control is switched to $u = -1$.



(a) $u = -1$ to $u = 1$ case.



(b) $u = 1$ to $u = -1$ case.

The minimum-time control strategy terminating at the origin as a function of the state variables can be expressed as

$$u = \begin{cases} -1, & \text{if } x_1 < 0 \text{ and } x_2 \geq -(x_1 - 1)^2 + 1; \\ -1, & \text{if } x_1 \geq 0 \text{ and } x_2 > (x_1 + 1)^2 - 1; \\ 0, & \text{if } x_1 = x_2 = 0; \\ 1, & \text{if } x_1 \leq 0 \text{ and } x_2 < -(x_1 - 1)^2 + 1; \\ 1, & \text{if } x_1 > 0 \text{ and } x_2 \leq (x_1 + 1)^2 - 1. \end{cases}$$