

**Exam#2**  
**In-Class Portion**  
**90 minutes**

Dec. 06, 1990

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1. Consider a control system described by

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - u(t) \\ \dot{x}_2(t) &= -2x_2(t) - 2u(t),\end{aligned}$$

where  $x_1$  and  $x_2$  are the state variables, and  $u$  is the control variable. Assuming that  $|u(t)| \leq 1$ , determine the trajectory of the state variables under a control that will transfer the states from  $\begin{bmatrix} -4 & 6 \end{bmatrix}^T$  to  $\begin{bmatrix} -2 & 0 \end{bmatrix}^T$  in minimum time. Plot the trajectory, but do not compute the switching time instance(s).

2. Consider a discrete-time linear system described by

$$\begin{aligned}x(k+1) &= x(k) \\ z(k+1) &= x(k+1) + v(k+1),\end{aligned}$$

where  $x$  and  $z$  are the state and the output variables, respectively,  $x(0)$  is a gaussian random variable, and  $v(k+1)$  is a zero-mean gaussian stochastic process with  $E[v(k+1)v(j+1)] = r\delta_{kj}$ . Assuming that  $x(0)$  and  $v(k+1)$  are independent, determine the filtered estimate of the state  $\hat{x}(k+1 | k+1)$  in terms of  $p_0 = p(0 | 0) = E[x^2(0)]$ ,  $r$ ,  $\hat{x}(k | k)$ , and  $z(k+1)$ . HINT: Use the second formulation of the Kalman gain equations; and to determine  $p(k | k)$  in terms of  $p(0 | 0)$ , first obtain  $1/p(k+1 | k+1)$  in terms of  $1/p(k | k)$ .

3. A discrete-time linear system is described by

$$\begin{aligned}x(k+1) &= 2x(k) + \omega(k) \\ z(k+1) &= x(k+1) + v(k+1),\end{aligned}$$

where  $x$  and  $z$  are the state and the output variables, respectively; and the system and the output noises are zero-mean, independent gaussian stochastic processes with covariances  $E[\omega(k)\omega(j)] = 2\delta_{kj}$  and  $E[v(k+1)v(j+1)] = \delta_{kj}$ . Determine the filtered estimate  $\hat{x}(k+1 | k+1)$  and the predicted estimate  $\hat{x}(k+1 | k)$  for  $k = 0$  and  $1$ ; if the initial condition is a zero-mean gaussian random variable that is independent of the other stochastic processes,  $E[x^2(0)] = p(0) = 1/2$ ,  $z(1) = 2$ , and  $z(2) = 3$ .

4. A constant speed motor with a very large motor time constant can be described by

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix}$$

$$z(t) = [1 \ 0] \begin{bmatrix} \theta(t) \\ \omega(t) \end{bmatrix} + v(t)$$

after normalization, where  $\theta$  and  $\omega$  are the state variables, and  $z$  is the output variable. Assuming that the initial state vector  $\begin{bmatrix} \theta(0) & \omega(0) \end{bmatrix}^T$  is a zero-mean gaussian random vector that is independent of the output noise, with covariance

$$P(0) = \begin{bmatrix} p_0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the noise is a zero-mean gaussian stochastic process with  $E[v(t)v^T(\tau)] = r\delta(t-\tau)$ , determine the kalman gain matrix  $K(t)$  in terms of  $p_0$  and  $r$ . HINT: Some elements of  $P(t | t)$  can be guessed from the problem definition.

#1

$$\dot{x}_1 = -x_1 - u$$

$$\dot{x}_2 = -2x_2 - 2u$$

$$|u| \leq 1$$

$$x(0) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}, x(T) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

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in minimum time

$$H = 1 + \lambda_1 \{x_1 - u\} + \lambda_2 \{-2x_2 - 2u\}$$

$$\text{OPT. EQN: } \dot{x} = H_x \Rightarrow \dot{x}_1 = -x_1 - u$$

$$\ddot{x}_2 = -2x_2 - 2u$$

$$\dot{\lambda} = -H_x \Rightarrow \dot{\lambda}_1 = -(-\lambda_1) = \lambda_1,$$

$$\dot{\lambda}_2 = -(-2\lambda_2) = 2\lambda_2$$

$$\begin{aligned} H^* \leq H &\Rightarrow 1 + \lambda_1^* \{-x_1^* - u^*\} + \lambda_2^* \{-2x_2^* - 2u^*\} \\ &\leq 1 + \lambda_1^* \{-x_1^* - u\} + \lambda_2^* \{-2x_2^* - 2u\} \\ &-(\lambda_1^* + 2\lambda_2^*) u^* \leq -(\lambda_1^* + 2\lambda_2^*) u \end{aligned}$$

$$u^* = \begin{cases} 1 & : -(\lambda_1 + 2\lambda_2) < 0 \\ -1 & : -(\lambda_1 + 2\lambda_2) > 0 \end{cases}$$

$$\text{let } u = \pm 1 \Rightarrow \dot{x}_1 = -x_1 \mp 1 \Rightarrow x_1 = c_1 e^{-t} \mp 1$$

$$\dot{x}_2 = -2x_2 \mp 2 \Rightarrow x_2 = c_2 e^{-2t} \mp 1$$

$$\text{Solve for } t : \bar{e}^t = \frac{x_1 \mp 1}{c_1}, e^{-2t} = \frac{(x_1 \mp 1)^2}{c_1^2}$$

$$\text{so } x_2 = \frac{c_2}{c_1^2} [x_1 \mp 1]^2 \mp 1$$

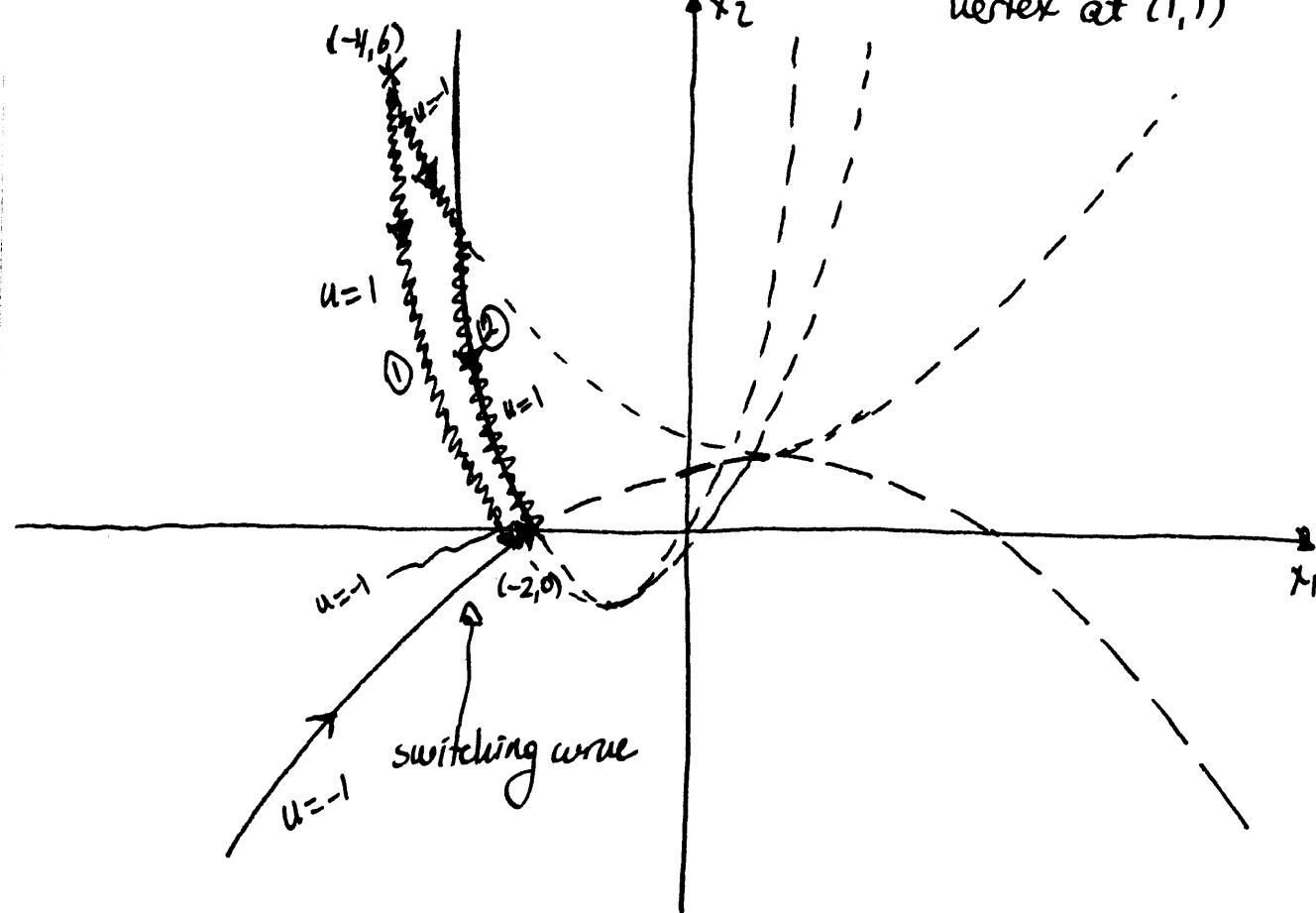
$$\text{or } x_2 = c_5 [x_1 \pm 1]^2 - 1$$

i.e. for  $u=1$  :  $x_2 = c_5 + (x_1 + 1)^2 - 1$

parabola with vertex at  $(-1, -1)$

$$u=-1 ; x_2 = c_5 - (x_1 - 1)^2 + 1$$

parabola with vertex at  $(1, 1)$



So two solns are possible

$$\textcircled{1} \quad u=1 \text{ on } x_2 = c_5 + (x_1 + 1)^2 - 1 : c_5 \text{ from } (-4, 6)$$

$$u=-1 \text{ on } x_2 = c_5 - (x_1 - 1)^2 + 1 : c_5 \text{ from } (2, 0)$$

i.e.  $u=1$  on  $x_2 = \frac{7}{9}(x_1 + 1)^2 - 1 \Rightarrow$  switch at  
 $u=-1$  on  $x_2 = -\frac{1}{9}(x_1 - 1)^2 + 1 \Rightarrow$  the intersection

$$\frac{7}{9}(x_1 + 1)^2 - 1 = -\frac{1}{9}(x_1 - 1)^2 + 1$$

$$4x_1^2 + 6x_1 - 5 = 0, x_1 = \frac{-3 \pm \sqrt{29}}{4}$$

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$$\text{In our case } x_1 = \frac{-3 - \sqrt{29}}{4}$$

$$\text{and } x_2 = \frac{33 - 7\sqrt{29}}{72} \text{ after subs.}$$

so  $t=0$

$$x(0) = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \xrightarrow{u=1} x(t_1) = \begin{bmatrix} -2.0963 \\ -0.0652 \end{bmatrix} \xrightarrow{u=-1} x(T) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\downarrow$

$$\begin{aligned} x_1 &= c_1 e^{-t} - 1 \\ x_2 &= c_2 e^{-2t} - 1 \end{aligned}$$

$\downarrow$

$$\begin{aligned} x_1 &= -3e^{-t} - 1 \\ x_2 &= 7e^{-2t} - 1 \end{aligned}$$

$\downarrow$

Given  $x(t_1)$

$$\Rightarrow t_1 = 1.00668$$

$\uparrow$

$\downarrow$

$$\begin{aligned} x_1 &= c_1 e^{-t} + 1 \\ x_2 &= c_2 e^{-2t} + 1 \end{aligned}$$

$\downarrow$

$$\begin{aligned} x_1 &= -8.4730 e^{-t} + 1 \\ x_2 &= -7.9769 e^{-2t} + 1 \end{aligned}$$

$\downarrow$

Given  $x(T)$

$$\Rightarrow T = 1.03827$$

(2)  $u = -1$  on  $x_2 = c_5 - (x_1 - 1)^2 + 1$  :  $c_5$  from  $(-4, 6)$

$$u = 1 \text{ on } x_2 = c_5 + (x_1 + 1)^2 - 1 ; c_5 \text{ from } (-2, 0)$$

i.e.  $u = -1$  on  $x_2 = \frac{1}{5}(x_1 - 1)^2 + 1$  ) switch at  
 $u = 1$  on  $x_2 = (x_1 + 1)^2 - 1$  the intersection

$$\frac{1}{5}(x_1 - 1)^2 + 1 = (x_1 + 1)^2 - 1$$

$$2x_1^2 + 6x_1 - 3 = 0 , x_1 = \frac{-3 \pm \sqrt{15}}{2}$$

$$\text{In our case } x_1 = \frac{-3 - \sqrt{15}}{2}$$

$$\text{and } x_2 = \frac{6 + \sqrt{15}}{2} \text{ after subs.}$$

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so  $t=0$ 

$$x(0) = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \xrightarrow{u=-1} x(t_1) = \begin{bmatrix} -3.4365 \\ 4.9365 \end{bmatrix} \xrightarrow{u=1} x(T) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\uparrow$

$$\begin{aligned} x_1 &= c_1 e^{-t} + 1 \\ x_2 &= c_2 e^{-2t} + 1 \end{aligned}$$

$\downarrow$

$$\begin{aligned} x_1 &= -5e^{-t} + 1 \\ x_2 &= 5e^{-2t} + 1 \end{aligned}$$

$\downarrow$

Given  $x(t_1)$   
 $\Rightarrow t_1 = 0.1196$

$\downarrow$

$$\begin{aligned} x_1 &= c_1 e^{-t} - 1 \\ x_2 &= c_2 e^{-2t} - 1 \end{aligned}$$

$\downarrow$

Given  $x(T)$   
 $\Rightarrow T = 1.0101$

Therefore the second possibility gives the minimum time,  
and

$$t=0 \quad t=0.1196 \quad t=1.0101$$

$$x(0) = \begin{bmatrix} -4 \\ 6 \end{bmatrix} \xrightarrow{u=-1} x(0.1196) = \begin{bmatrix} -3.4365 \\ 4.9365 \end{bmatrix} \xrightarrow{u=1} x(1.0101) = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

#2

$$x(k+1) = x(k)$$

$$z(k+1) = x(k+1) + v(k+1)$$

 $x(0)$  Gaussian

$$E\{x(0)\} = 0$$

 $v(k+1)$  Gaussian

$$E\{v(k+1)\} = 0$$

$$E\{v(k+1)v(j+1)\} = r \delta_{jk}$$

$$E\{x(0)v(k+1)\} = 0$$

We had  $\hat{x}(k+1/k+1) = \hat{x}(k+1/k) + K(k+1) [z(k+1) - H(k+1) \hat{x}(k/k)]$

or  $\hat{x}(k+1/k+1) = \hat{x}(k/k) + K(k+1) [z(k+1) - \hat{x}(k/k)]$

Also  $P(k+1) = P(k+1/k+1) H^T(k+1) R^{-1}(k+1)$

or  $K(k+1) = P(k+1/k+1) \cdot \frac{1}{r} = \frac{P(k+1/k+1)}{r}$

$$P(k+1/k+1) = P(k+1/k) - P(k+1/k) H^T(k+1) \left[ H(k) P(k+1/k) H^T(k+1) + R(k) \right]^{-1} H(k+1) P(k+1/k)$$

or  $P(k+1/k+1) = P(k+1/k) - P(k+1/k) \left[ P(k+1/k) + r \right]^{-1} P(k+1/k)$   
 $= P(k+1/k) - \frac{P^2(k+1/k)}{P(k+1/k) + r}$   
 $= \frac{r P(k+1/k)}{P(k+1/k) + r} = \frac{1}{\frac{1}{r} + \frac{1}{P(k+1/k)}}$

$$P(k+1/k) = Q(k+1/k) P(k/k) Q^T(k+1/k) + R(k) P^T(k+1/k)$$

or  $P(k+1/k) = P(k/k)$

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$$\text{So } p(k+1/k+1) = \frac{1}{r + \frac{1}{p(k/k)}}$$

$$\text{or } \frac{1}{p(k+1/k+1)} = \frac{1}{r} + \frac{1}{p(k/k)}$$

$$\frac{1}{p(k/k)} = \frac{k}{r} + \frac{1}{p(0/0)} = \frac{k}{r} + \frac{1}{p_0} = \frac{1+kp_0/r}{p_0}$$

$$p(k/k) = \frac{p_0}{1+\frac{kp_0}{r}}$$

$$\text{i.e. } K(k+1) = \frac{p_0}{r+(k+1)p_0}$$

$$\text{and } \hat{x}(k+1/k+1) = \hat{x}(k/k) + \frac{p_0}{r+(k+1)p_0} [z(k+1) - \hat{x}(k/k)]$$

#3

$$x(k+1) = 2x(k) + w(k)$$

$$z(k+1) = x(k+1) + v(k+1)$$

$$w(k) \text{ Gaussian } E\{w(k)\} = 0$$

$$E\{w(k)w(j)\} = 2\delta_{kj}$$

$$v(k+1) \text{ Gaussian } E\{v(k+1)\} = 0$$

$$E\{v(k+1)v(j+1)\} = \delta_{kj}$$

$$x(0) \text{ Gaussian } E\{x(0)\} = 0$$

$$E\{x^2(0)\} = \frac{1}{2}$$

$$E\{w(k)v(j+1)\} = 0$$

$$E\{w(k)x(0)\} = 0$$

$$E\{v(k+1)x(0)\} = 0$$

$$\text{and } z(1) = 2$$

$$z(2) = 3$$

$$\hat{x}(k+1/k+1) = \underline{\Phi}(k+1, k) \hat{x}(k/k) + K(k+1) [z(k+1) - H(k+1) \underline{\Phi}(k+1, k) \hat{x}(k/k)]$$

$$\text{or } \hat{x}(k+1/k+1) = 2\hat{x}(k/k) + K(k+1) [z(k+1) - 2\hat{x}(k/k)]$$

$$\hat{x}(k+1/k) = \underline{\Phi}(k+1, k) \hat{x}(k/k)$$

$$\text{or } \hat{x}(k+1/k) = 2\hat{x}(k/k)$$

$$\text{Also } K(k+1) = P(k+1/k) H^T(k+1) [H(k+1) P(k+1/k) H^T(k+1) + R(k+1)]^{-1}$$

$$\text{or } K(k+1) = P(k+1/k) [P(k+1/k) + 1]^{-1}$$

$$= \frac{P(k+1/k)}{P(k+1/k) + 1}$$

$$P(k+1/k) = \Phi(k+1/k)P(k/k) \Phi^T(k+1/k) + \Gamma(k+1/k)Q(k)\Gamma^T(k+1/k)$$

or  $P(k+1/k) = 4P(k/k) + 2$

$$P(k+1/k+1) = [\Phi - K(k+1)H(k+1)]P(k+1/k)$$

$$\text{or } P(k+1/k+1) = [1 - K(k+1)]P(k+1/k)$$

$$\text{so for } k=0 : \quad p(0/0) = p(0) = \frac{1}{2} \quad \text{given}$$

$$K(1) = \frac{4}{4+1} = \frac{4}{5}$$

$$p(1/1) = [1 - \frac{4}{5}]4 = \frac{4}{5}$$

$$\text{for } k=1 : \quad p(2/1) = 4 \cdot \frac{4}{5} + 2 = \frac{26}{5}$$

$$K(2) = \frac{\frac{26}{5}}{\frac{26}{5} + 1} = \frac{26}{31}$$

$$\text{Then } \hat{x}(1/1) = 2 \cdot 0 + \frac{4}{5}[2 - 2 \cdot 0] = \frac{8}{5}$$

$$\hat{x}(2/2) = 2 \cdot \frac{8}{5} + \frac{26}{31}[3 - 2 \cdot \frac{8}{5}] = \frac{94}{31}$$

$$\text{and } \hat{x}(1/0) = 2 \cdot 0 = 0$$

$$\hat{x}(2/1) = 2 \cdot \frac{8}{5} = \frac{16}{5}$$

#4

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$$

$$z = [1 \ 0] \begin{bmatrix} \theta \\ \omega \end{bmatrix} + v$$

$\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$  zero mean Gaussian with  $P(0) = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$

and  $v$  zero mean Gaussian with  $E[v(t)v^T(t)] = r\delta(t-z)$

and  $\begin{bmatrix} \theta(0) \\ \omega(0) \end{bmatrix}$  and  $v$  are indep.

$$\begin{aligned} K(t) &= P(t/t) H_{(t)}^T R^{-1}(t) \\ &= \frac{1}{r} P(t/t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

$$\text{where } \dot{P} = AP + PA^T - PH^T R^{-1} H P + GQG^T$$

$$\text{i.e. } \begin{bmatrix} \dot{P}_1 & \dot{P}_2 \\ \dot{P}_2 & \dot{P}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$-\boxed{\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} \frac{1}{r} \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}}$$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \quad \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$

$$\text{so } \dot{P}_1 = P_2 + P_2 - \frac{1}{r} P_1^2$$

$$\dot{P}_2 = P_3 + 0 - \frac{1}{r} P_1 P_2$$

$$\dot{P}_3 = 0 + 0 - \frac{1}{r} P_2^2$$

with

$$\rho_{1(0)} = \rho_0$$

$$\begin{aligned} \rho_{2(0)} &= 0 \\ \rho_{3(0)} &= 0 \end{aligned} \quad \left. \right\} \Rightarrow$$

initially  $\rho_2$  and  $\rho_3$  are zero, i.e.  
uncorrelated, and deterministic,  
since there is no further  
correlation,  $\rho_2(t/t) = \rho_3(t/t) = 0$

thus satisfies the  $\dot{\rho}_2$  and  $\dot{\rho}_3$  equations.

then  $\dot{\rho}_1 = -\frac{1}{r}\rho_1^2$  where  $\rho_{1(0)} = \rho_0$

$$\frac{d\rho_1}{dt} = -\frac{1}{r}\rho_1^2$$

$$\int \frac{d\rho_1}{\rho_1^2} = -\frac{1}{r} \int dt + C$$

$$-\frac{1}{\rho_1} = -\frac{1}{r}t + C_1$$

$$\rho_1 = \frac{1}{t/r + C_2}$$

since  $\rho(0) = \rho(0/0) = \rho_0$

$$\rho_0 = \frac{1}{C_2} \quad \text{or} \quad C_2 = \frac{1}{\rho_0}$$

then

$$\rho_1(t/t) = \frac{1}{t/r + 1/\rho_0}$$

$$\text{Therefore } K(t) = \frac{1}{r} \begin{bmatrix} \frac{1}{t/r + 1/\rho_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{t + r/\rho_0} \\ 0 \end{bmatrix} \quad t \geq 0$$