An Unexpected Maximum in a Family of Rectangles

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Introduction The genesis for this paper was Problem 749 from the Macalester College problem-of-the-week series:

MACALESTER PROBLEM 749. *Given a square $A_1A_2A_3A_4$ and a point P inside the square. The lengths of PA_1, PA_2, and PA_3 are 4, 3, and $\sqrt{10}$, respectively. What is the length of PA_4?*

The reader is encouraged to try to solve the problem now, before proceeding. In our solution to this problem we realized that $A_1A_2A_3A_4$ need not be a square. This observation led to the study of families of rectangles that satisfy hypotheses like those of Problem 749, and ultimately to this paper.

A quick solution to Problem 749 is provided by Theorem 1, which is a special case of Feuerbach's Relation (see **[4]** or **[6]**).

THEOREM 1. *If P is any point in the plane of rectangle $A_1A_2A_3A_4$, and if a_i is the distance from P to A_i, then $\sum_{i=1}^{4}(-1)^i a_i^2 = 0$.*

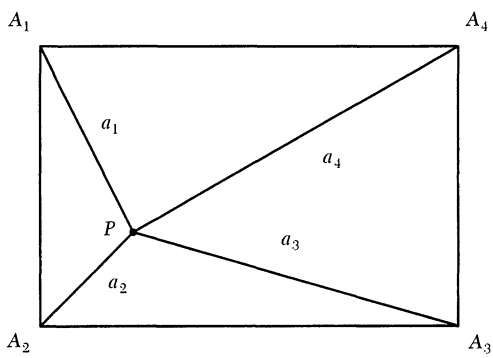

FIGURE 1
$a_1^2 + a_3^2 = a_2^2 + a_4^2$.

Thus, an ordered triple of distances from a point P to three consecutive vertices of a rectangle uniquely determines the *distance* from P to the fourth vertex, but does not uniquely determine the *rectangle*. For instance, let a_1, a_2, a_3, and a_4 be given with $\sum_{i=1}^{4}(-1)^i a_i^2 = 0$. If vertex A_2 of the rectangle is fixed at the origin and P is on the circle Γ with center A_2 and radius a_2, then vertices A_3 and A_1 will be on the x- and y-axes, respectively, and vertex A_4 will be determined by the positions of A_3 and A_1. As P moves around Γ, the rectangle changes. See FIGURE 2.

We used *The Geometer's Sketchpad* to explore various properties of the rectangles. The software allowed us to keep a_1, a_2, a_3, and a_4 fixed while moving the point P around the circle Γ, thus showing how the rectangles change. A natural question arises: *When is the area of the rectangle an extremum?* Although the perimeter is not

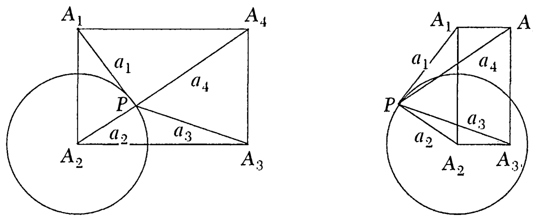

FIGURE 2
Two rectangles with the same a_i's.

constant, a reasonable-sounding guess was that the extrema occur when the rectangle is a square. Alas, experiments using the area computation feature of *The Geometer's Sketchpad* indicated that this guess was incorrect. We solve the area-optimization problem in the last section.

Another natural question is to describe the locus of A_4 under the construction given above. The "trace locus" feature of *The Geometer's Sketchpad* shows, when a_2 is the smallest distance, that this locus is a closed curve that resembles a guitar pick. See FIGURE 3. If a_2 is not less than both a_3 and a_1, then the rectangle does not exist for P on certain arcs of the circle Γ, and the locus of A_4 will be a disconnected curve. In the next section we discuss the equations for the locus of A_4.

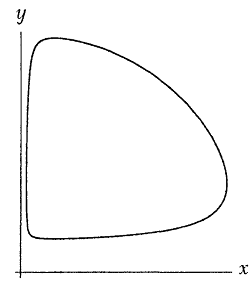

FIGURE 3
A 'guitar pick' traced by vertex A_4.

In our search of the literature we discovered that problems related to Problem 749 have a long history. For example, an old problem in geometry is to construct a specified kind of triangle when the distances from a point in the plane of the triangle to its vertices are given. Geometry problem 151 in the June-July 1901 *American Mathematical Monthly* **[3]** gave the distances from a point to three corners of a square, which is equivalent to specifying a right isosceles triangle. Baker **[1]** says the problem in **[3]** is a variation of Rutherford's problem, in which the triangle is equilateral. When the triangle is equilateral and the distances are 3, 4, and 5, Rabinowitz **[5]** and Gregorac **[2]** call the problem the 3-4-5 puzzle. Many other references may be found in **[5]**. Walter **[7]** considers distances of 3, 4, and 5 to three consecutive vertices of a rectangle. Several generalizations have been studied, such as letting the distances be a, b, and c (**[3]**, **[5]**, **[7]**), allowing the point from which the distances are measured to be above (or below) the plane of the triangle (**[7]**), replacing the triangle by a polygon (**[5]**), and replacing the triangle by a simplex in $\mathbb{R}^n$ with $n+1$ vertices (**[2]**).

The locus of A_4 Let a_1, a_2, a_3, and a_4 be fixed, with $\sum_{i=1}^{4}(-1)^i a_i^2 = 0$. Relabel the points and distances, if necessary, so that a_2 is the minimum of the distances with A_2 fixed at the origin. Let the coordinates of A_3 be $(x, 0)$, and the coordinates of A_1 be $(0, y)$. Since P is on the circle Γ, which has radius a_2, the coordinates of P are

$(a_2\cos\theta,\ a_2\sin\theta)$, where θ is the angle measured from the positive x-axis to A_2P. Since A_3 is on the circle with center P and radius a_3,

$$(x - a_2\cos\theta)^2 + (a_2\sin\theta)^2 = a_3^2.$$

One solution to this quadratic in x is

$$x = x(\theta) = a_2\cos\theta + \sqrt{a_3^2 - a_2^2\sin^2\theta}\,. \tag{1}$$

Similarly, since A_1 is on the circle with center P and radius a_1, we get

$$y = y(\theta) = a_2\sin\theta + \sqrt{a_1^2 - a_2^2\cos^2\theta}\,. \tag{2}$$

These are parametric equations for the locus of A_4, with parameter θ in $[0, 2\pi)$. Taking the positive sign in front of the radicals guarantees that the rectangle will be in the first quadrant.

However, equations (1) and (2) are only one of four pairs of parametric equations for the locus of A_4. The other three are determined by using the negative square root in one or both equations. The four curves are given by

$$C_{mn}:\begin{cases} x = a_2\cos(\theta) + (-1)^{m+n}\sqrt{a_3^2 - a_2^2\sin^2(\theta)} \\ y = a_2\sin(\theta) + (-1)^{n+1}\sqrt{a_1^2 - a_2^2\cos^2(\theta)} \end{cases} \qquad m,n = 1,2.$$

More possibilities arise if we do not require a_2 to be the minimum distance. Again, there are four cases: (a) $a_2 < a_1$ and $a_2 < a_3$, (b) $a_2 < a_1$ and $a_2 > a_3$, (c) $a_2 > a_1$ and $a_2 < a_3$, (d) $a_2 > a_1$ and $a_2 > a_3$. In cases (b), (c), and (d), each of the curves C_{mn} is disconnected. For example, in case (b) the curve C_{11} is given by equations (1) and (2). The quantity under the radical in (1) is negative for two θ-intervals, $\theta_1 < \theta < \theta_2$, and $\theta_3 < \theta < \theta_4$. Further, $x(\theta_2) = -x(\theta_1)$ and $x(\theta_4) = -x(\theta_3)$. Thus the curve "jumps" to the second quadrant when $\theta = \theta_2$ and "jumps back" to the first quadrant when $\theta = \theta_4$. Similar behavior occurs in case (c) with (2), and in case (d) with both (1) and (2). FIGURE 4 shows all sixteen possible curves C_{mn} for cases (a)–(d), with the plot styles for the curves as shown in FIGURE 4a.

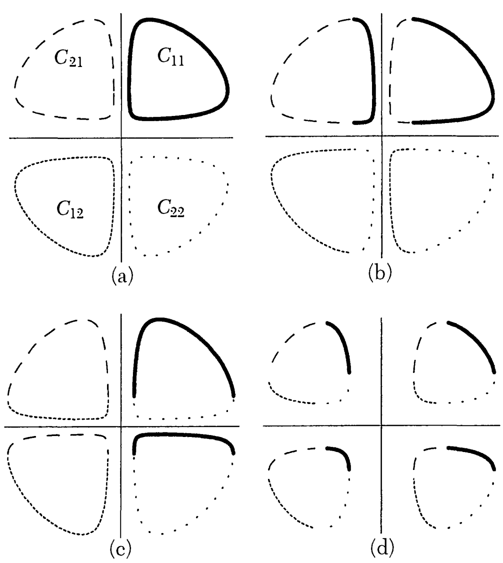

FIGURE 4

$a_i \in \{1.3, 1.5, 1.8\}$.

(a) $a_2 < a_3$ and $a_2 < a_1$ (b) $a_2 > a_3$ and $a_2 < a_1$

(c) $a_2 < a_3$ and $a_2 > a_1$ (d) $a_2 > a_3$ and $a_2 > a_1$

Optimizing the area The area of the rectangle can be expressed in terms of θ. Let

$$a(\theta) = \left(a_2 \cos\theta + \sqrt{a_3^2 - a_2^2 \sin^2\theta}\right) \cdot \left(a_2 \sin\theta + \sqrt{a_1^2 - a_2^2 \cos^2\theta}\right).$$

From (1) and (2) it follows that the area of the rectangle is $A(\theta) = |a(\theta)|$ whenever $a(\theta)$ is real-valued. Note that we are not assuming that a_2 is the least of the distances and that, because of the symmetry of the four curves in each part of FIGURE 4, we can restrict ourselves to C_{11}.

Insight into where $A(\theta)$ is optimal may be obtained by using *Mathematica* or another computer algebra system to plot $A(\theta)$, $x(\theta)$, and $y(\theta)$ on the same axes for $0 \le \theta \le 2\pi$. These graphs show that $A(\theta)$ is not necessarily largest when the rectangle is a square, as our earlier experiments with *The Geometer's Sketchpad* indicated. In FIGURE 5a (in which $a_1 = 7$, $a_2 = 3$, and $a_3 = 5$) the extrema for the area

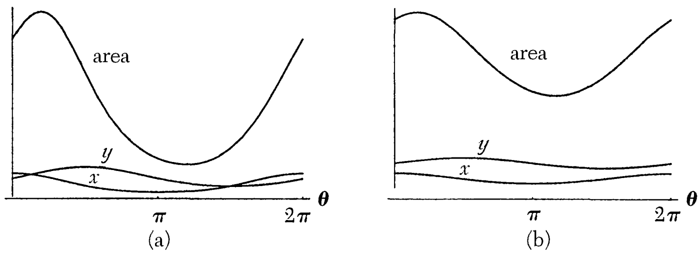

FIGURE 5
Extrema need not correspond to squares, which may not exist.

curve do not correspond to intersections of the x and y curves. In fact, the rectangle may *never* become a square, as shown in FIGURE 5b (in which $a_1 = 7$, $a_2 = 1$, and $a_3 = 4$), where the graphs of x and y do not intersect.

To optimize $A(\theta)$, we first find and simplify $a'(\theta)$.

$$a'(\theta) = \left(\frac{a_2}{\sqrt{a_1^2 - a_2^2 \cos^2\theta}\sqrt{a_3^2 - a_2^2 \sin^2\theta}}\right) \cdot \left(\cos\theta\sqrt{a_3^2 - a_2^2 \sin^2\theta} - \sin\theta\sqrt{a_1^2 - a_2^2 \cos^2\theta}\right) \cdot a(\theta).$$

The chain rule then gives

$$A'(\theta) = \left(\frac{a_2 \cdot \left(\cos\theta\sqrt{a_3^2 - a_2^2 \sin^2\theta} - \sin\theta\sqrt{a_1^2 - a_2^2 \cos^2\theta}\right)}{\sqrt{a_1^2 - a_2^2 \cos^2\theta}\sqrt{a_3^2 - a_2^2 \sin^2\theta}}\right) \cdot |a(\theta)|.$$

If a_2 exceeds either a_1 or a_3 there will be intervals in which $A(\theta)$ and $A'(\theta)$ are not real valued. This is illustrated in FIGURE 6 (in which $a_1 = 5.5$, $a_2 = 6$, and $a_3 = 5$). However, the values of $A(\theta)$ and the values of the one-sided derivatives $A'_{\pm}(\theta)$ at the endpoints of each of these intervals agree. Hence, $A(\theta)$ may be viewed as being a differentiable function on the interval $[0, 2\pi]$ minus the intervals where $A(\theta)$ is not real valued. Thus, the only critical points occur when $\cos\theta\sqrt{a_3^2 - a_2^2 \sin^2\theta} = \sin\theta\sqrt{a_1^2 - a_2^2 \cos^2\theta}$. This condition simplifies to $\tan\theta = \pm a_3/a_1$. The negative is an extraneous solution, so the only critical points of interest occur for θ such that

$$\tan\theta = a_3/a_1. \tag{3}$$

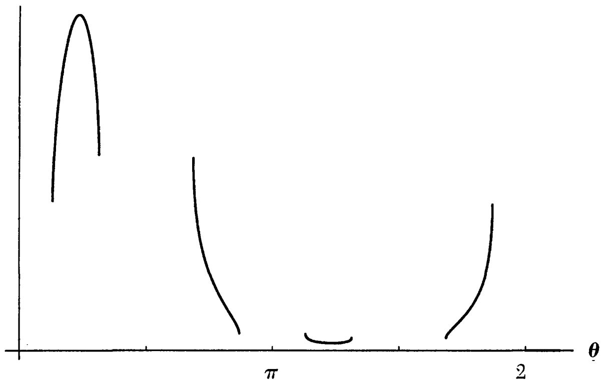

FIGURE 6

Graph of $A(\theta)$ is discontinuous if a_2 is greater than a_1 or a_3.

If $0 < \theta < \arctan(a_3/a_1)$ then $a_3/\sqrt{a_1^2 + a_3^2} < \sin\theta$ and $\cos\theta < a_1/\sqrt{a_1^2 + a_3^3}$. This observation implies that $A'(\theta) > 0$ for θ in $(0, \arctan(a_3/a_1))$. Similar reasoning shows that if $\arctan(a_3/a_1) < \theta < \pi/2$ then $A'(\theta)$ is negative. Hence, if θ satisfies (3) and lies in the first quadrant, then the corresponding rectangle has maximum area. Analogous arguments show that if θ satisfies (3) and lies in the third quadrant, the corresponding rectangle has minimum area.

To gain geometric insight into why the extrema occur when $\tan\theta = a_3/a_1$, construct the auxiliary rectangle $B_1A_2B_3B_4$ with B_3 on the x-axis, B_1 on the y-axis, $A_2B_3 = a_1$, and $A_2B_1 = a_3$. See FIGURE 7. Then the angle between the positive x-axis and A_2B_4 is the same as the angle for which the area is maximum. Further, rectangle $B_1A_2B_3B_4$ can be used to determine where P must be on Γ for a rectangle to exist for a given set of distances a_i. In the first quadrant, P must lie on the arc of Γ which is inside $B_1A_2B_3B_4$. Outside the auxiliary rectangle, either $PA_3 > a_3$ or $PA_1 > a_1$. Symmetric conditions hold in the other quadrants. The condition $\tan\theta = a_3/a_1$ for maximum area can also be thought of as describing the angle for the limiting position of P as $a_2 \to \sqrt{a_3^2 + a_1^2}$ with $a_3^2 + a_1^2$ fixed. In the special case of $a_4 = 0$, the family of rectangles has exactly four congruent members, corresponding to $\tan\theta = \pm a_3/a_1$.

The above discussion is summarized in the following theorem.

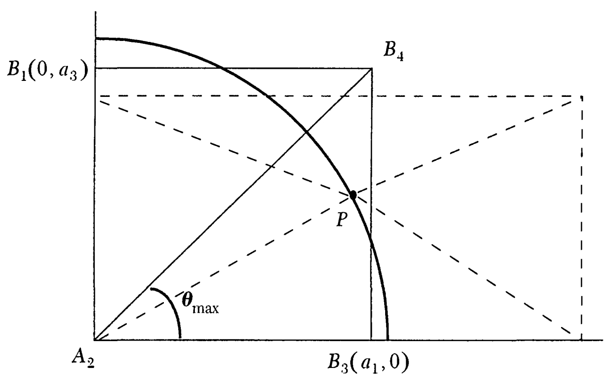

FIGURE 7

P must be on the arc inside $B_1A_2B_3B_4$.

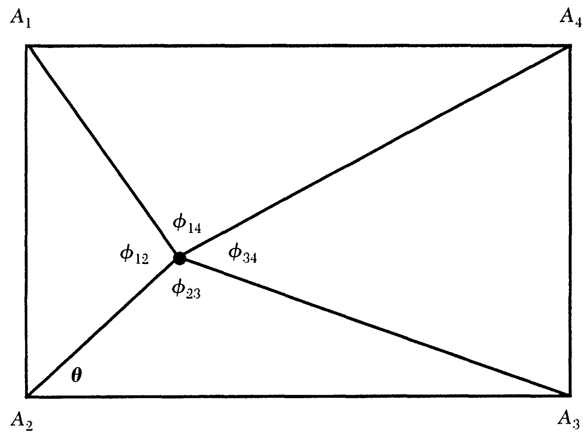

FIGURE 8
The angles around P.

THEOREM 2. *If the distances a_1, a_2, a_3, and a_4 satisfy $\sum_{i=1}^{4}(-1)^i a_i^2 = 0$ and determine a family of rectangles as described above, then the rectangle with maximum area occurs when the angle θ is in the first quadrant and $\tan\theta = a_3/a_1$. The rectangle with minimum area occurs when the angle θ is in the third quadrant and $\tan\theta = a_3/a_1$.*

Additional results are given as corollaries; proofs are left to the reader. Denote the angles subtended from P by the sides of the rectangle by $\phi_{ij} = \angle A_i P A_j$. See FIGURE 8.

COROLLARY 2.1. *When the area of the rectangle is maximum, $\phi_{12} + \phi_{34} = \phi_{23} + \phi_{14} = \pi$.*

Question: What is the relationship among the ϕ_{ij} when the area is minimum?

COROLLARY 2.2. *The maximum area is given by $a_1 a_3 + a_2 a_4$, and the minimum area is given by $|a_1 a_3 - a_2 a_4|$. Thus, any set $\{a_1, a_2, a_3, a_4\}$ of distances from P to consecutive vertices of a family of rectangles may be permuted, subject to $\sum_{i=1}^{4}(-1)^i a_i^2 = 0$, without changing the maximum and minimum areas.*

A second approach to finding these extrema is to use Lagrange multipliers. The vectors $\overrightarrow{A_2 P}$, $\overrightarrow{PA_3}$, and $\overrightarrow{PA_1}$ can be combined to give

$$\overrightarrow{A_2 A_3} = \overrightarrow{A_2 P} + \overrightarrow{PA_3}; \qquad \overrightarrow{A_2 A_1} = \overrightarrow{A_2 P} + \overrightarrow{PA_1}.$$

Note that the (signed) area of the rectangle is the product of the first component of $\overrightarrow{A_2 A_3}$ with the second component of $\overrightarrow{A_2 A_1}$, and that both the second component of $\overrightarrow{A_2 A_3}$ and the first component of $\overrightarrow{A_2 A_1}$ must be 0 since A_3 and A_1 are on the coordinate axes. The problem becomes: Minimize

$$a(\theta, \phi_{23}, \phi_{12}) = \left(a_2 \cos\theta - a_3 \cos(\theta + \phi_{23})\right)\left(a_2 \sin\theta + a_1 \sin(\phi_{12} - \theta)\right),$$

subject to the constraints

$$a_2 \sin\theta = a_3 \sin(\theta + \phi_{23}) \quad \text{and} \quad a_2 \cos\theta = a_1 \cos(\phi_{12} - \theta).$$

Details are left to the readers (or their calculus classes).

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Counting Integer Triangles

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Introduction How many triangles with integer sides have a given perimeter? This elementary counting problem came up in a geometry class. As we soon found out, the answer has been known for a long time. In rederiving that answer for ourselves we found the Internet and *Maple* to be valuable tools. This note describes our experiences in finding the number t_n of congruence classes of triangles with integer sides summing to n, which we will just call *integer triangles*.

Data stage Although we guessed that the answer to such a classical problem would be known, we did not have a good idea where to look for it. So we began our attack on the problem in the most primitive possible way, constructing small triangles. Obviously $1, 1, 1$ is the smallest integer triple for sides of a triangle, followed by $2, 2, 1$, then $2, 2, 2$ and two triangles of perimeter 7: $3, 2, 2$ and $3, 3, 1$. Continuing in this way, we found one integer triangle of perimeter 8, three of perimeter 9, two of perimeter 10, and four of perimeter 11.

These data suggested to us two possible integer sequences. One would list perimeters of all integer triangles, and begin $3, 5, 6, 7, 7, 8, 9, 9, 9, 10, 10, 11, 11, 11, 11$. The other would list the number of integer triangles of each perimeter, and so would begin $1, 0, 1, 1, 2, 1, 3, 2, 4$. With these sequences in hand, we turned immediately to our battered copy of N.J.A. Sloane's classic *Handbook of Integer Sequences*. Unfortunately, we came up empty. (We would not have come up empty if we had owned the new *Encyclopedia of Integer Sequences* by N.J.A. Sloane and S. Plouffe.) So we turned to our computer, submitting the sequence to Sloane (at AT&T Research) by e-mail (address: `sequences@research.att.com`; subject: none; message: `lookup 1 0 1 1 2 1 3 2 4`). We were quickly informed that the second of our sequences was in fact *Alcuin's sequence*, the coefficients in the power series expansion of

$$\frac{x^3}{(1-x^2)(1-x^3)(1-x^4)}. \tag{1}$$